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State Change, Quantum Probability, and Information in Operational Phase-Space Measurement

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State change, quantum probability, and information gain in the operational phase-space measurement are formulated by means of positive operator-valued measure (POVM) and operation. The properties of the operational POVM and its marginal POVM which yield the quantum probability distributions of the measurement outcomes obtained by the operational phase-space measurement are investigated. The Naimark extension of the operational POVM can be expressed in terms of the relative-position states and the relative-momentum states in the extended Hilbert space. An observable quantity measured in the operational phase-space measurement becomes a fuzzy or unsharp observable. The state change of a physical system caused by the operational phase-space measurement is described by the operation which is obtained explicitly for the position and momentum measurements and for the simultaneous measurement of position and momentum. Using the results, the entropy change of the measured physical system and the information gain in the operational phase-space measurement are investigated. It is found that the average value of the entropy change is equal to the Shannon mutual information extracted from the outcomes exhibited by the measurement apparatus.

1. INTRODUCTION

The operational phase-space measurement (Wódkiewicz, 1984, 1986, 1987; Burak and Wódkiewicz, 1992; Bužek *et al.*, 1995a, b; Englert and Wódkiewicz, 1995), which is closely related to the simultaneous quantum measurement of position and momentum (Arthurs and Kelly, 1965; Busch, 1985; Stenholm, 1992) and to the quantum measurement on fuzzy or unsharp observables (Prugovečki, 1973, 1974, 1975, 1976a, b, 1977; Twareque Ali and

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Emch, 1974; Twareque Ali and Doebner 1976; Twareque Ali and Prugovečki, 1977a, b; Morato, 1977), is formulated in terms of quantum probability distributions which are functions of position and momentum variables defined on the quantum phase space. The quantum probability distribution can be expressed in terms of the smoothed Wigner function, which is the convolution of two Wigner functions (O'Connell and Wigner, 1981; Smith, 1988; Lalović *et al.*, 1992; Orlowski and Wünsche, 1993), or as the convolution of the *P*-function and *Q*-function, which are quasi-probability distributions in the phase space. Such smoothing is ascribed to the finite accuracy of a measurement apparatus and to the fuzziness or unsharpness of a measured physical quantity. Unlike the Wigner function, the smoothed Wigner function is nonnegative and can be interpreted as the probability distribution of position and momentum that is defined on phase space analogous to the one in the classical statistical mechanics (Mayer and Mayer, 1977).

The phase-space description of quantum mechanics is also useful for investigating quantum chemical systems (Torres-Vega, 1993a, b; Torres-Vega and Frederick, 1990, 1993; Harriman, 1994; Wlodarz, 1994; Moller *et al.*, 1997). Furthermore, an uncertainty relation, called the operational uncertainty relation (Wódkiewicz, 1987), is established in the operational phase-space measurement. The minimum value of the operational uncertainty is twice that of the usual uncertainty of position and momentum variables. This is because the uncertainty caused by the measurement apparatus as well as the intrinsic uncertainty in the quantum state of a physical system enter in our observation. It is shown that the operational phase-space probability distribution can be derived within the framework of the relative-state formulation of quantum systems (Ban, 1991, 1993a, 1996) that was originally constructed to introduce a quantum mechanical phase operator (Ban, 1992a, b, 1993b, 1994a, b). Moreover, the new interpretation of the scalar product in the Hilbert space of a quantum system considered by several authors (Prugovečki, 1982; O'Connell and Rajagopal, 1982; Aharonov *et al.*, 1981) is closely related to the concept of the operational phase-space measurement. The operational approach to optical homodyne detection has recently been considered (Banaszek and Wódkiewicz, 1997).

Quantum measurement is mathematically described in term of a positive operator-valued measure (POVM), also called an effect, in the most general way (Davies, 1976; Helstrom, 1976; Holevo, 1982; Kraus, 1983; Busch *et al.*, 1995; Ozawa, 1984, 1993; Peres, 1993). POVM includes a projection-valued measure as a special case that describes the standard quantum measurement considered by von Neumann (1955), referred to as the first-kind measurement. The quantum probability of some measurement outcome is calculated by the POVM and the statistical operator of a measured quantum state. The change of the quantum state of a physical system caused by quantum measurement is described by an operation (or a completely positive instrument) (Davies, 1976; Kraus, 1983; Busch *et al.*, 1995; Ozawa, 1984, 1993), which is expressed as a superoperator (Fano, 1957; Crawford, 1958; Prigogine *et al.*, 1973; Schmutz, 1978; Umezawa *et al.*, 1982; Umezawa, 1993). An operation is related to a POVM through a quantum probability.

So far, the operational phase-space quantum measurement has been formulated in an empirical and intuitive way (Wódkiewicz, 1984, 1986, 1987; Burak and Wódkiewicz, 1992; Bužek *et al.*, 1995a, b; Englert and Wódkiewicz, 1995). Furthermore, although the quantum probability distributions of the measurement outcomes were obtained, the state change caused by the operational phase-space measurement has not been considered in detail. Therefore, using the POVM and the operation, we will investigate the operational phase-space measurement and obtain the change of the quantum state caused by the effect of the measurement in a systematic way.

An information-theoretical approach to quantum measurement has recently attracted much attention since it provides a basis for quantum information, communication, and computation (Bendjaballah *et al.*, 1991; Belavkin *et al.*, 1995; Hirota *et al.*, 1997). Therefore it is important to investigate the entropy change of a physical system and the information gain extracted from measurement outcomes in the operational phase-space measurement. In this paper, we will consider the measurement entropy (Ballan *et al.*, 1986) and the relation to the Shannon mutual information or the mean information content (Shannon, 1948a, b; Brillouin, 1956; Majernik, 1970, 1973; Cover and Thomas, 1991) in the operational phase-space measurement. For this purpose, we have to obtain the POVM and the operation for the operational phase-space measurement. It will be shown that the average value of the entropy change of a physical system in the operational phase-space measurement of position or momentum is equal to the Shannon mutual informations extracted from the outcomes exhibited by the measurement apparatus.

The remainder of this paper is organized as follows. In Section 2, we briefly summarize the operational phase-space measurement in a convenient way for our purpose (Wódkiewicz, 1984, 1986, 1987, Burak and Wódkiewicz, 1992; Bužek *et al.*, 1995a, b; Englert and Wódkiewicz, 1995). There we give several examples of operational phase-space probability distributions. In Section 3, we obtain the operational POVM that describes the operational phase-space measurement. Furthermore, we investigate the Naimark extensions of the operational POVM and show that the Naimark extensions are expressed in terms of the relative-position states and the relative-momentum states (Ban, 1993a, b, 1996; Hongi-yi and Klauder, 1994; Hongi-yi and Xiong, 1995; Hongi-yi and Yue, 1996). Using the results, we investigate the marginal POVM and the observable quantity in the operational phase-space measurement. The quantity measured in the operational phase-space measurement becomes a fuzzy or unsharp observable. In Section 4, we obtain the operation that describes the state change of the physical system caused by the operational phase-space measurement. There we consider position and momentum measurements and the simultaneous measurement of position and momentum. To this end, we apply the superoperator method or thermofield dynamics (Fano, 1957; Crawford, 1958; Prigogine *et al.*, 1973; Schmutz, 1978; Umezawa *et al.*, 1982; Umezawa, 1993) to express the results in a simple form. In Section 5, we consider the entropy change of the physical system and the information gain in the operational phase-space measurement of position and momentum. It is shown that the average value of the entropy change is equal to the Shannon mutual information obtained from the measurement outcomes shown by the measurement apparatus. Furthermore, the simultaneous measurement of position and momentum is also considered. In this case, we use the Q-function and the Wehrl entropy (Wehrl, 1978, 1979) as the probability distribution of position and momentum and the entropy. In Section 6, we summarize our results.

2. OPERATIONAL PHASE-SPACE MEASUREMENT

2.1. Operational Probability Distribution

In this section, we summarize the operational phase-space measurement (Wódkiewicz, 1984, 1986, 1987; Burak and Wódkiewicz, 1992; Bužek *et al.*, 1995a, b; Englert and Wódkiewicz, 1995) in a convenient way for our purpose. Suppose that we observe a physical system in quantum state $\hat{\rho}$, where $\hat{\rho}$ is a statistical operator defined on the Hilbert space $\mathcal{H}$ of the physical system. To observe the physical system, we must use a measurement apparatus that is prepared in quantum state $\hat{\sigma}_a$ defined on the Hilbert space $\mathcal{H}_a$ of the measurement apparatus. Of course, the quantum state $\hat{\sigma}_a$ should be appropriate for the measurement that we carry out. Let $\hat{\sigma}$ be a statistical operator defined on the Hilbert space $\mathcal{H}$ of the physical system, which has the same property as that of the statistical operator $\hat{\sigma}_a$ of the measurement apparatus (see Section 4). In the operational phase-space measurement, the quantum state described by the statistical operator $\hat{\sigma}$ is referred to as a quantum-filter state or a quantum-ruler state (Bužek *et al.*, 1995a, b).

In the operational phase-space measurement, to observe the physical system, we compare the quantum state $\hat{\rho}$ of the physical system with the quantum-filter state $\hat{\sigma}$. To do this, we first transport the quantum-filter state $\hat{\sigma}$ to the measured physical system in the phase space, and then we investigate the overlap between the quantum states $\hat{\rho}$ and $\hat{\sigma}$. This transportation in the phase space is given by the unitary transformation $\hat{D}(r, k)\hat{\sigma}\hat{D}^{\dagger}(r, k)$, where the pair (r, k) represents the amount of the transportation in the two-dimensional phase space and the unitary operator $\hat{D}(r, k)$ is the displacement operator given by

$$\hat{D}(r, k) = \exp[i(k\hat{x} - r\hat{p})] \qquad (2.1)$$
$$= \exp[\mu\hat{a} - \mu^*\hat{a}^\dagger] = \hat{D}(\mu)$$

Here the operators $\hat{x}$ and $\hat{p}$ are the canonical position and momentum operators of the physical system, satisfying the canonical commutation relation $[\hat{x}, \hat{p}] = i$ ($\hbar = 1$), and the operators $\hat{a}$ and $\hat{a}^\dagger$ are bosonic annihilation and creation operators, which satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, where the operators $\hat{a}$ and $\hat{a}^\dagger$ are related to $\hat{x}$ and $\hat{p}$ by the relations

$$\hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}}, \qquad \hat{a}^\dagger = \frac{\hat{x} - i\hat{p}}{\sqrt{2}} \qquad (2.2)$$

Furthermore, the complex parameter μ is defined by

$$\mu = \frac{r + ik}{\sqrt{2}} \qquad (2.3)$$

In this paper, we will confine ourselves to considering two-dimensional phase space. The generalization to phase space of higher dimensionality is straightforward.

Therefore the operational phase-space probability distribution $\mathcal{W}(r, k)$ (Wódkiewicz, 1986; Bužek *et al.*, 1995a; Ban, 1996) is given by

$$\mathcal{W}(r, k) = \frac{1}{2\pi} \mathrm{Tr}[\hat{\rho}\hat{D}(r, k)\hat{\sigma}\hat{D}^\dagger(r, k)] \qquad (2.4)$$

where Tr stands for the trace operation over the Hilbert space of the physical system. It is easy to verify that the operational phase-space probability distribution $\mathcal{W}(r, k)$ is nonnegative and normalized as

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \mathcal{W}(r, k) = 1 \qquad (2.5)$$

The quantity $\mathcal{W}(r, k)\Delta r\, \Delta k$ represents the probability that the measured values of the phase-space variables r and k belong to the ranges $(r, r + \Delta r)$ and $(k, k + \Delta k)$, where Δr and Δk are infinitesimal quantities. The meaning of the phase-space variables r and k will be considered later. The operational phase-space probability distribution $\mathcal{W}(r, k)$ given by equation (2.4) is also called the propensity of the quantum state (Wódkiewicz, 1984, 1986).

The operational phase-space probability distribution $\mathcal{W}(r, k)$ can be expressed in terms of well-known phase-space functions such as the P-function and Q-function (Husimi function). To do this, we introduce the phase-space function $P_\rho(\alpha; s)$ of the quantum state $\hat{\rho}$ (Cahill and Glauber, 1969a, b; Agarwal and Wolf, 1970a–c),

$$P_\rho(\alpha; s) = \mathrm{Tr}[\hat{\rho}\hat{T}(\alpha; s)] \tag{2.6}$$

with

$$\hat{T}(\alpha; s) = \frac{1}{\pi}\int_{\xi\in\mathbb{R}^2} d^2\xi\, \hat{D}(\xi; s) \exp[\alpha\xi^* - \alpha^*\xi] \tag{2.7}$$

$$\hat{D}(\xi; s) = \exp\left[\xi\hat{a}^\dagger - \xi^*\hat{a} + \frac{1}{2} s|\xi|^2\right] \tag{2.8}$$

where $d^2\xi = d(\mathrm{Rec}\ \xi)\, d(\mathrm{Im}\ \xi)$ and $\mathrm{Re}(\xi)$ [$\mathrm{Im}(\xi)$] stands for taking the real (imaginary) part of complex variable ξ. In equation (2.7), $\mathbb{R}$ stands for the whole real axis. The phase-space function $P_\rho(\alpha; s)$ is called the s-ordered quasi-probability distribution, since $P_\rho(\alpha; s)$ is normalized as $\int_{\alpha\in\mathbb{R}^2} d^2\alpha\, P_\rho(\alpha; s) = 1$, but in not nonnegative. The statistical operator $\hat{\rho}$ is expressed in terms of $P_\rho(\alpha; s)$ and $\hat{T}(\alpha; s)$,

$$\hat{\rho} = \frac{1}{\pi}\int_{\alpha\in\mathbb{R}^2} d^2\alpha\, P_\rho(\alpha; -s)\hat{T}(\alpha; s) \tag{2.9}$$

Using the s-ordered quasi-probability distributions, we can express the operational phase-space probability distribution $\mathcal{W}(r, k)$ as

$$\mathcal{W}(r, k) = \frac{1}{2\pi^2}\int_{\alpha\in\mathbb{R}^2} d^2\alpha\, P_\rho(\alpha + \mu; s)P_\sigma(\alpha; -s) \tag{2.10}$$

where we set $\mu = (r + ik)/\sqrt{2}$, and $P_\sigma(\alpha; s)$ is the s-ordered quasi-probability function of the quantum-filter state $\hat{\sigma}$.

It is easy to see from the definitions that the phase-space functions $P_\rho(\alpha) = (1/\pi)P_\rho(\alpha; 1)$, $W_\rho(\alpha) = (1/\pi)P_\rho(\alpha; 0)$, and $Q_\rho(\alpha) = (1/\pi)P_\rho(\alpha; -1)$ are, respectively, the P-function (Glauber, 1963a, b; Sudarshan, 1963), the Wigner function (Wigner, 1932; Hillery *et al.*, 1984), and the Q-function (Husimi function) (Husimi, 1940; Kano, 1965; Mehta and Sudarshan, 1965) of the quantum state $\hat{\rho}$, which are given by

$$P_\rho(\alpha) = \frac{1}{\pi} P_\rho(\alpha; 1) = \frac{1}{\pi^2}\int_{\xi\in\mathbb{R}^2} d^2\xi\, \mathrm{Tr}[\hat{\rho} e^{\xi\hat{a}^\dagger} e^{-\xi^*\hat{a}}] e^{\alpha\xi^* - \alpha^*\xi} \tag{2.11}$$

$$W_\rho(\alpha) = \frac{1}{\pi} P_\rho(\alpha; 0) = \frac{1}{\pi^2}\int_{\xi\in\mathbb{R}^2} d^2\xi\, \mathrm{Tr}[\hat{\rho} e^{\xi\hat{a}^\dagger - \xi^*\hat{a}}] e^{\alpha\xi^* - \alpha^*\xi} \tag{2.12}$$

$$Q_\rho(\alpha) = \frac{1}{\pi} P_\rho(\alpha; -1) = \frac{1}{\pi^2}\int_{\xi\in\mathbb{R}^2} d^2\xi\, \mathrm{Tr}[\hat{\rho} e^{-\xi^*\hat{a}} e^{\xi\hat{a}^\dagger}] e^{\alpha\xi^* - \alpha^*\xi} \tag{2.13}$$

Thus, using these functions, we obtain the expressions for the operational phase-space probability distribution $\mathcal{W}(r, k)$ as follows:

$$\begin{aligned}\mathcal{W}(r, k) &= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, P_\rho(q + r, p + k) Q_\sigma(q, x) \\ &= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, Q_\rho(q + r, p + k) P_\sigma(q, x) \\ &= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, W_\rho(q + r, p + k) W_\sigma(q, x) \qquad (2.14)\end{aligned}$$

where we set $F(q, p) = F(\alpha)$ $(F = P, W, Q)$ with $\alpha = (q + ip)/\sqrt{2}$. The last expression in this equation is the smoothed Wigner function (O'Connell and Wigner, 1981; Smith, 1988; Lalović *et al.*, 1992; Orlowski and Wünsche, 1993).

In particular, when the quantum-filter state is vacuum, namely, $\hat{\sigma} = |0\rangle\langle 0|$, the operational phase-space probability distribution $\mathcal{W}(r, k)$ becomes

$$\mathcal{W}(r, k) = \exp\left[\frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial k^2}\right)\right] W_\rho(r, k) \qquad (2.15)$$

On the other hand, when the quantum state of the physical system and the quantum-filer state are both pure, setting $\hat{\rho} = |\psi\rangle\langle\psi|$ and $\hat{\sigma} = |\phi\rangle\langle\phi|$ yields the following expression for the operational phase-space probability distribution $\mathcal{W}(r, k)$:

$$\begin{aligned}\mathcal{W}(r, k) &= \frac{1}{2\pi} \left| \langle\psi|\hat{D}(r, k)|\phi\rangle \right|^2 \\ &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} dx\, \psi(x + r)\phi(x) \exp[-ikx] \right|^2 \qquad (2.16)\end{aligned}$$

which is the same form that was used for considering a new interpretation of the scalar product in the Hilbert space of the physical system (Prugovečki, 1982; O'Connell and Rajagopal, 1982; Aharonov *et al.*, 1981).

2.2. Properties of the Probability Distribution

We now consider the properties of the operational phase-space probability distribution $\mathcal{W}(r, k)$. We first investigate the average values of the phase-space variables r and k and their fluctuations. For the sake of simplicity, we first assume that the quantum-filter state is the vacuum state $\hat{\sigma} = |0\rangle\langle 0|$. Then it is found from equation (2.4) that

$$\langle r \rangle \equiv \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, r\,{}^{\sigma}W(r, k) = \mathrm{Tr}[\hat{x}\hat{\rho}] \equiv \langle \hat{x} \rangle_{\rho} \tag{2.17}$$

$$\langle k \rangle \equiv \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, k\,{}^{\sigma}W(r, k) = \mathrm{Tr}[\hat{p}\hat{\rho}] \equiv \langle \hat{p} \rangle_{\rho} \tag{2.18}$$

Thus the average values of the phase-space variables r and k are equal to those of the position and momentum of the physical system in the quantum state $\hat{\rho}$. The fluctuations are given by

$$(\Delta r)^2 \equiv \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk (r - \langle r \rangle)^2\,{}^{\sigma}W(r, k) = (\Delta_{\rho}\hat{x}) + \frac{1}{2} \tag{2.19}$$

$$(\Delta k)^2 \equiv \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk (k - \langle k \rangle)^2\,{}^{\sigma}W(r, k) = (\Delta_{\rho}\hat{p})^2 + \frac{1}{2} \tag{2.20}$$

where we have defined $(\Delta_{\rho}\hat{x})^2 = \langle(\hat{x} - \langle\hat{x}\rangle_{\rho})^2\rangle_{\rho}$ and $(\Delta_{\rho}\hat{p})^2 = \langle(\hat{p} - \langle\hat{p}\rangle_{\rho})^2\rangle_{\rho}$. This result indicates that the fluctuations of the phase-space variables r and k are larger by 1/2 than the intrinsic fluctuations of the position and momentum of the physical system in the quantum state $\hat{\rho}$. The increase of the fluctuation is ascribed to the unsharpness or the finite accuracy of the measurement apparatus. It is clear that the value of 1/2 is the magnitude of the vacuum fluctuation of the measurement apparatus.

For an arbitrary quantum-filter state $\hat{\sigma}$, we obtain the average values and fluctuations of the phase-space variables r and k (Ban, 1996),

$$\langle r \rangle = \langle \hat{x} \rangle_{\rho} + c_x, \qquad \langle k \rangle = \langle \hat{p} \rangle_{\rho} + c_p \tag{2.21}$$

$$(\Delta r)^2 = (\Delta_{\rho}\hat{x})^2 + (\Delta_{\sigma}\hat{x})^2 \tag{2.22}$$

$$(\Delta k)^2 = (\Delta_{\rho}\hat{p})^2 + (\Delta_{\sigma}\hat{p})^2 \tag{2.23}$$

where c_x and c_p are the parameters that correspond to the phase-space coordinate of the measurement apparatus and we may set $c_x = c_p = 0$ by appropriately choosing the origin of the coordinate system of the phase space. The additional fluctuations $(\Delta_{\sigma}\hat{x})^2$ and $(\Delta_{\sigma}\hat{p})^2$ in the quantum-filter state $\hat{\sigma}$ are given by

$$(\Delta_{\sigma}\hat{x})^2 = \mathrm{Tr}[\hat{x}^2\hat{\sigma}] - \{\mathrm{Tr}[\hat{x}\hat{\sigma}]\}^2 \tag{2.24}$$

$$(\Delta_{\sigma}\hat{p})^2 = \mathrm{Tr}[\hat{p}^2\hat{\sigma}] - \{\mathrm{Tr}[\hat{p}\hat{\sigma}]\}^2 \tag{2.25}$$

It is considered that additional fluctuations are caused by the finite accuracy of the measurement apparatus. Therefore we obtain the operational uncertainty relation of the phase-space variables (Wódkiewicz, 1987),

$$(\Delta r)^2(\Delta r)^2 \geq [(\Delta_\rho \hat{x})(\Delta_\rho \hat{p}) + (\Delta_\sigma \hat{x})(\Delta_\sigma \hat{p})]^2 \geq 1 \tag{2.26}$$

where we set $\hbar = 1$. It is found from the above consideration that the phase-space variables r and k represent the position and momentum of the measured physical system, which includes the effect of the measurement apparatus.

Next we consider the marginal distributions of the operational phase-space probability distribution $\mathcal{W}(r, k)$, which are given by

$$\mathcal{W}_r(r) = \int_{-\infty}^{\infty} dk\, \mathcal{W}(r, k) \tag{2.27}$$

$$\mathcal{W}_k(k) = \int_{-\infty}^{\infty} dr\, \mathcal{W}(r, k) \tag{2.28}$$

Substituting (2.4) or (2.14) into these equations, we can express the marginal distributions $\mathcal{W}_r(r)$ and $\mathcal{W}_k(k)$ as

$$\mathcal{W}_r(r) = \int_{-\infty}^{\infty} dx\, f(x - r)\langle x|\hat{\rho}|x\rangle \tag{2.29}$$

$$\mathcal{W}_k(k) = \int_{-\infty}^{\infty} dk\, g(p - k)(p|\hat{\rho}|p) \tag{2.30}$$

where the functions $f(x)$ and $g(p)$ are given by

$$f(x) = \langle x|\hat{\sigma}|x\rangle, \qquad g(p) = (p|\hat{\sigma}|p) \tag{2.31}$$

In equations (2.29)–(2.31), state vectors $|x\rangle$ and $|p)$ represent the eigenstates of the canonical position and momentum operators, that is, $\hat{x}|u\rangle = u|u\rangle$, $\hat{x}|u) = i\partial|u)/\partial u$ and $\hat{p}|u\rangle = -i\partial|u\rangle/\partial u$, $\hat{p}|u) = u|u)$. It is seem from (2.29) and (2.30) that the function $f(x)$ or $g(p)$ is the filter function that determines the accuracy of the measurement apparatus in the position or momentum measurement on the physical system. Therefore, it is found that the quantum-filter state $\hat{\sigma}$ characterizes the measurement apparatus in the operational phase-space measurement. The marginal distributions given by (2.29) and (2.29) appear in the fuzzy space formulation of quantum mechanics (Prugovečki, 1973, 1974, 1975, 1976a, b, 1977; Twareque Ali and Emch, 1974; Twareque Ali and Doebner, 1976; Twareque Ali and Prugovečki, 1977a, b; Morato, 1977).

When the quantum-filter state is the squeezed-vacuum state with real squeezing parameter γ (Yuen, 1976), namely,

$$\hat{\sigma} = |\gamma\rangle\langle\gamma|, \qquad |\gamma\rangle = \exp\left[\frac{1}{2}\gamma(\hat{a}^{\dagger 2} - \hat{a}^2)\right]|0\rangle \tag{2.32}$$

the filter functions $f(x)$ and $g(p)$ are given by

$$f(x) = \frac{1}{e^{\gamma}\sqrt{\pi}}\exp\left(-\frac{x^2}{e^{2\gamma}}\right) \tag{2.33}$$

$$g(p) = \frac{1}{e^{-\gamma}\sqrt{\pi}}\exp\left(-\frac{p^2}{e^{-2\gamma}}\right) \tag{2.34}$$

Thus we see that if $\gamma > 0$ ($\gamma < 0$), the measurement of the momentum (position) is more accurate than that of the position (momentum). In particular, we can approximate $\mathcal{W}_r(r) \approx \langle r|\hat{\rho}|r\rangle$ for $-\gamma >> 1$ and $\mathcal{W}_k(k) \approx (k|\hat{\rho}|k)$ for $\gamma >> 1$. Therefore, if $-\gamma >> 1$ ($\gamma >> 1$), the additional fluctuation caused by the measurement apparatus does not come in the position (momentum) measurement on the physical system. But we cannot obtain $\mathcal{W}_r(r) \approx \langle r|\hat{\rho}|r\rangle$ and $\mathcal{W}_k(k) \approx (k|\hat{\rho}|k)$ simultaneously, which violates the uncertainty relation of the measurement apparatus.

2.3. Examples of the Operational Probability Distribution

Before closing this section, we give the operational phase-space probability distributions $\mathcal{W}(r, k)$ for the several quantum-filter states $\hat{\sigma}$. We first consider the vacuum filter state $\hat{\sigma} = |0\rangle\langle 0|$. In this case, the operational phase-space probability distribution becomes the Q-function (Husimi function) (Husimi, 1940; Kano, 1965; Mehta and Sudarshan, 1965) of the quantum state $\hat{\rho}$ of the physical system,

$$\mathcal{W}(r, k) = \frac{1}{2\pi}\langle\mu|\hat{\rho}|\mu\rangle \tag{2.35}$$

where $|\mu\rangle$ is the coherent state with complex amplitude $\mu = (r + ik)/\sqrt{2}$. The phase marginal distribution $\mathcal{W}_\phi(\theta)$ of equation (2.35) can be used for investigating the phase properties of the photon (Burak and Wódkiewicz, 1992),

$$\mathcal{W}_\phi(\theta) = \int_0^\infty dR\, R\mathcal{W}(r, k) \tag{2.36}$$

where $R = \sqrt{r^2 + k^2}$ and $\tan\theta = \dfrac{k}{r}$.

When we choose the Fock state $|n\rangle\langle n|$ as the quantum-filter state, we obtain the operational phase-space probability distribution,

$$\mathcal{W}(r, k) = \frac{1}{2\pi} \langle \mu, n|\hat{\rho}|\mu; n\rangle \tag{2.37}$$

where $|\mu; n\rangle = \hat{D}(\mu)|n\rangle$ is the displaced number state (Boiteux and Levelut, 1973; Mahran and Satyanarayana, 1986; de Oliveira *et al.*, 1990). In particular, when the measured physical system is prepared in the coherent state $|\alpha\rangle$ with $\alpha = (q + ip)/\sqrt{2}$, equation (2.37) is calculated to be

$$\mathcal{W}(r, k) = \frac{1}{2^{n+1}n!\pi} [(r - q)^2 + (k - p)^2]^n \exp\left[-\frac{1}{2}(r - q)^2 - \frac{1}{2}(k - p)^2 \right] \tag{2.38}$$

which yields the average values and fluctuations of the phase-space variables, $\langle r\rangle = q$, $\langle k\rangle = p$ and $(\Delta r)^2 = (\Delta k)^2 = 1 + n$. Thus, the higher excited quantum-filter state gives the lower measurement accuracy.

Next we consider the thermal state $\hat{\rho}_{\text{th}}$ as the quantum-filter state,

$$\hat{\rho}_{\text{th}} = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}}\right)^n |n\rangle\langle n| \tag{2.39}$$

where $\bar{n}$ is the average value of the thermal photon number. In this case, the operational phase-space probability probability distribution is obtained by averaging equation (2.37) with respect to n by means of the probability $p(n) = \bar{n}^n/(1 + \bar{n})^{1+n}$,

$$\mathcal{W}(r, k) = \frac{1}{2\pi(1 + \bar{n})} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}}\right)^n \langle \mu, n|\hat{\rho}|\mu; n\rangle \tag{2.40}$$

When the measured physical system is in the coherent state $|\alpha\rangle$, we obtain the following Gaussian distribution:

$$\mathcal{W}(r, k) = \frac{1}{2\pi(1 + \bar{n})} \exp\left[-\frac{(r - g)^2 + (k - p)^2}{2(1 + \bar{n})} \right] \tag{2.41}$$

where we set $\alpha = (q + ip)/\sqrt{2}$. Then we have the average values and fluctuations of the phase-space variables, $\langle r\rangle = q$, $\langle k\rangle = p$, and $(\Delta r)^2 = (\Delta k)^2 = 1 + \bar{n}$. It is reasonable that the measurement accuracy becomes lower as the thermal noise of the measurement apparatus increases.

The marginal distributions of position for the operational phase-space probability distributions, (2.38) and (2.41), become

$$\mathcal{W}_r(x) = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{n} \frac{(2m)!}{2^{m+n}(m!)^2(n-m)!} (x-q)^{2(n-m)} \exp\left[-\frac{1}{2}(x-q)^2 \right] \tag{2.42}$$

$$\mathcal{W}_r(x) = \frac{1}{\sqrt{2\pi(1+\bar{n})}} \exp\left[-\frac{(x-q)^2}{2(1+\bar{n})} \right] \tag{2.43}$$

On the other hand, the intrinsic position probability of the physical system prepared in the coherent state $|\alpha\rangle$ is given by

$$P(x) = |\langle\alpha|x\rangle|^2 = \frac{1}{\sqrt{\pi}} \exp[-(x-q)^2] \tag{2.44}$$

The position probability distributions given by equations (2.42)–(2.44) are plotted in Fig. 1. The figure show the characteristic feature of the operational phase-space probability, where the effect of the measurement apparatus changes the shape of the probability distribution.

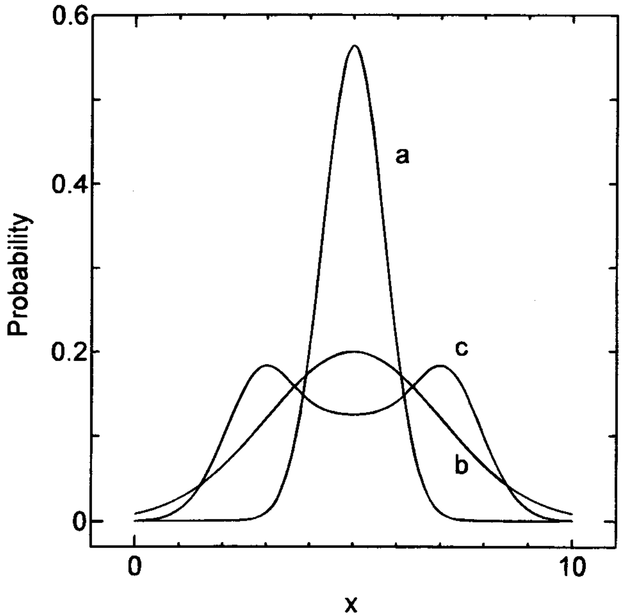

Fig. 1. Probability distribution of the position of the physical system in the coherent state $|\alpha\rangle$ with $\sqrt{2}$ Re $\alpha = q = 5$. (a) The intrinsic position probability of the physical system [equation (2.44)], (b) the operational position probability for the thermal filter state with $\bar{n} = 2$ [equation (2.43)], and (c) the operational position probability for the number filter state with $n = 2$ [equation (2.42)].

Furthermore, when the quantum filter is in the coherent state $|\beta\rangle$, the operational phase-space probability distribution becomes the Q-function,

$$\mathcal{W}(r, k) = \frac{1}{2\pi} \langle \mu + \beta | \hat{\rho} | \mu + \beta \rangle \tag{2.45}$$

We finally consider the squeezed-vacuum state given by equation (2.32) as the quantum-filter state. Using the Wigner function $W_\rho(q, p)$ of the quantum state $\hat{\rho}$ of the physical system, we obtain the operational phase-space probability probability distribution,

$$\mathcal{W}(r, k) = \frac{1}{\pi} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp \exp[-e^{-2\gamma}(r - q)^2 - e^{2\gamma}(k - p)^2] W_\rho(q, p) \tag{2.46}$$

which is equivalent to the generalized antinormally ordered distribution function investigated by Lee (1995). For the coherent state $|\alpha\rangle$ with complex amplitude $\alpha = (q + ip)/\sqrt{2}$ of the physical system, equation (2.46) becomes

$$\mathcal{W}(r, k) = \frac{1}{\pi(e^{\gamma} + e^{-\gamma})} \exp\left[-\frac{(r - q)^2}{1 + e^{2\gamma}} - \frac{(k - p)^2}{1 + e^{-2\gamma}}\right] \tag{2.47}$$

which yields the average values and fluctuations of the phase-space variables, $\langle r \rangle = q$, $\langle k \rangle = p$, $(\Delta r)^2 = \frac{1}{2}(1 + e^{2\gamma})$, and $(\Delta k)^2 = \frac{1}{2}(1 + e^{-2\gamma})$.

3. QUANTUM PROBABILITY IN OPERATIONAL PHASE-SPACE MEASUREMENT

3.1. Operational Positive Operator-Valued Measure

The quantum probability that we obtain some measurement outcome when we perform a quantum measurement on a physical system can be expressed in terms of a positive operator-valued measure (POVM) and a statistical operator (Davies, 1976; Helstrom, 1976; Holevo, 1982; Kraus, 1983; Busch *et al.*, 1995; Ozawa, 1984, 1993; Peres, 1993). The quantum probability of measurement outcome x_j is given by $P(x_j) = \mathrm{Tr}[\hat{\Pi}(x_j)\hat{\rho}]$, where $\hat{\rho}$ is the statistical operator that describes the quantum state of the measured physical system and $\hat{\Pi}(x_j)$ is the POVM that describes the quantum measurement which yields the outcome x_j. The POVM $\hat{\Pi}(x_j)$ is a nonnegative Hermitian operator and constitutes a resolution of the identity,

$$\hat{\Pi}(x_j) \geq 0, \qquad \sum_j \hat{\Pi}(x_j) = \hat{I} \tag{3.1}$$

where $\hat{I}$ is an identity operator defined on the Hilbert space $\mathcal{H}$ of the physical system and the summation is taken over all possible measurement outcomes.

These relations ensure that the quantum probability $P(x_j)$ is nonnegative and normalizable.

To consider the POVM that describes the operational phase-space measurement, recall that the probability $\mathcal{W}(\Delta_r, \Delta_k)$ that the outcomes r and k of the operational phase-space measurement belong to the ranges Δ_r and Δ_k is obtained from equation (2.4),

$$\mathcal{W}(\Delta_r, \Delta_k) = \frac{1}{2\pi} \int_{r \in \Delta_r} dr \int_{k \in \Delta_k} dk \operatorname{Tr}[\hat{D}(r, k)\hat{\sigma}\hat{D}^\dagger(r, k)\hat{\rho}] \tag{3.2}$$

where Δ_r and Δ_k stand for intervals on the real axis. We refer to $\mathcal{W}(\Delta_r, \Delta_k)$ as the operational phase-space probability, and we easily find the following relations:

$$\mathcal{W}(\Delta_r, \Delta_k) \geq 0, \qquad \mathcal{W}(\mathbb{R}, \mathbb{R}) = 1, \qquad \mathcal{W}(\Delta_r, \emptyset) = \mathcal{W}(\emptyset, \Delta_k) = 0 \tag{3.3}$$

$$\mathcal{W}(\Delta_r^{(1)} \cup \Delta_r^{(2)}, \Delta_k) = \mathcal{W}(\Delta_r^{(1)}, \Delta_k) + \mathcal{W}(\Delta_r^{(2)}, \Delta_k) \tag{3.4}$$

$$\mathcal{W}(\Delta_r, \Delta_k^{(1)} \cup \Delta_k^{(2)}) = \mathcal{W}(\Delta_r, \Delta_k^{(1)}) + \mathcal{W}(\Delta_r, \Delta_k^{(2)}) \tag{3.5}$$

where $\mathbb{R}$ stands for the whole real axis and $\Delta_x^{(1)}$ and $\Delta_x^{(2)}$ $(x = r, k)$ are disjoined subsets of $\mathbb{R}$, namely, $\Delta_r^{(1)} \cap \Delta_r^{(2)} = \Delta_k^{(1)} \cap \Delta_k^{(2)} = \emptyset$.

The POVM $\hat{\Pi}(\Delta_r, \Delta_r)$ of the operational phase-space measurement, called the operational POVM, is the nonnegative Hermitian operator and the resolution of the identity such that it should satisfy the relation

$$\mathcal{W}(\Delta_r, \Delta_k) = \operatorname{Tr}[\hat{\Pi}(\Delta_r, \Delta_r)\hat{\rho}] \tag{3.6}$$

where $\hat{\rho}$ is the statistical operator of the physical system. Thus we obtain the following equality from (3.2) and (3.6):

$$\operatorname{Tr}[\hat{\Pi}(\Delta_r, \Delta_r)\hat{\rho}] = \frac{1}{2\pi} \int_{r \in \Delta_r} dr \int_{k \in \Delta_k} dk \operatorname{Tr}[\hat{D}(r, k)\hat{\sigma}\hat{D}^\dagger(r, k)\hat{\rho}] \tag{3.7}$$

Since this equality should hold for any statistical operator $\hat{\rho}$ of the physical system, the operational POVM $\hat{\Pi}(\Delta_r, \Delta_r)$ is given by

$$\hat{\Pi}(\Delta_r, \Delta_r) = \frac{1}{2\pi} \int_{r \in \Delta_r} dr \int_{k \in \Delta_k} dk \, \hat{D}(r, k)\hat{\sigma}\hat{D}^\dagger(r, k) \tag{3.8}$$

It is the easy task to see that the operational POVM $\hat{\Pi}(\Delta_r, \Delta_r)$ given by this equation satisfies the following relations:

$$\hat{\Pi}(\Delta_r, \Delta_k) \geq 0, \qquad \hat{\Pi}(\mathbb{R}, \mathbb{R}) = \hat{I}, \qquad \hat{\Pi}(\Delta_r, \emptyset) = \hat{\Pi}(\emptyset, \Delta_k) = 0 \tag{3.9}$$

$$\hat{\Pi}(\Delta_k^{(1)} \cup \Delta_r^{(2)}, \Delta_k) = \hat{\Pi}(\Delta_r^{(1)}, \Delta_k) + \hat{\Pi}(\Delta_r^{(2)}, \Delta_k) \tag{3.10}$$

$$\hat{\Pi}(\Delta_r, \Delta_k^{(1)} \cup \Delta_k^{(2)}) = \hat{\Pi}(\Delta_r, \Delta_k^{(1)}) + \hat{\Pi}(\Delta_r, \Delta_k^{(2)}) \tag{3.11}$$

$$[\hat{\Pi}(\Delta_r, \Delta_k)]^2 \neq \hat{\Pi}(\Delta_r, \Delta_k) \tag{3.12}$$

where $\Delta_r^{(1)}$ and $\Delta_r^{(2)}$ ($\Delta_k^{(1)}$ and $\Delta_k^{(2)}$) are disjoined subsets of $\mathbb{R}$. It is seen from the last relation that the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$ does not become a projection-valued measure. This indicates that the operational phase-space measurement (Wódkiewicz, 1984, 1986, 1987; Burak and Wódkiewicz, 1992; Bužek *et al.*, 1995a, b; Englert and Wódkiewicz, 1995) is an unsharp or fuzzy quantum measurement (Prugovečki, 1973, 1974, 1975, 1976a, b, 1977; Twareque Ali and Emch, 1974; Twareque Ali and Doebner, 1976; Twareque Ali and Prugovečki, 1977a, b; Morato, 1977). Expression (3.8) will be used in the next section to obtain the operation that describes the state change of the physical system caused by the operational phase-space measurement.

We assume that the quantum-filter state $\hat{\sigma}$ is a function of the annihilation and creation operators and we write it as $\hat{\sigma} = \mathcal{G}(\hat{a}, \hat{a}^\dagger)$. Then the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$ is expressed as

$$\hat{\Pi}(\Delta_r, \Delta_k) = \frac{1}{\pi} \int_{\mu \in \tilde{\Delta}_r \times \tilde{\Delta}_k} d^2\mu \; \mathcal{G}(\hat{a} - \mu, \hat{a}^\dagger - \mu^*) \tag{3.13}$$

where $\tilde{\Delta}_r$ and $\tilde{\Delta}_k$ are defined by $\tilde{\Delta}_r = \Delta_r/\sqrt{2}$ and $\tilde{\Delta}_k = \Delta_k/\sqrt{2}$, and $\mu \in \tilde{\Delta}_r \times \tilde{\Delta}_k$ means that $\mathrm{Re}\mu \in \tilde{\Delta}_r$ and $\mathrm{Im}\mu \in \tilde{\Delta}_k$. Furthermore, when there is the P-function of the quantum-filter state $\hat{\sigma}$, that is, $\hat{\sigma} = \int_{\alpha \in \mathbb{R}^2} d^2\alpha \, |\alpha\rangle P_\sigma(\alpha)\langle\alpha|$, we obtain the expression for the operational POVM,

$$\hat{\Pi}(\Delta_r, \Delta_k) = \frac{1}{\pi^2} \int_{\mu \in \tilde{\Delta}_r x \tilde{\Delta}_k} d^2\mu \int_{\alpha \in \mathbb{R}^2} d^2\alpha|\alpha\rangle P_\sigma(\alpha - \mu)\langle\alpha| \tag{3.14}$$

As the example, let us consider the thermal state as the quantum-filter state $\hat{\sigma}$,

$$\hat{\sigma} = \frac{1}{1 + \bar{n}} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^{\hat{a}^\dagger a}$$

$$= (1 - e^{-\theta}) \exp[-\theta \hat{a}^\dagger \hat{a}] \tag{3.15}$$

where the parameter θ is given by $\theta = \ln(1 + \bar{n}^{-1})$. Then, the operator $\hat{D}(r, k)\hat{\sigma}\, \hat{D}^\dagger(r, k)$ is calculated to be

$$\hat{D}(r, k)\hat{\sigma}\hat{D}^\dagger(r, k) = 2 \sinh\left(\frac{\theta}{2}\right) \exp\left[-\frac{1}{2}\, \theta(\hat{x} - r)^2 - \frac{1}{2}\, \theta(\hat{p} - k)^2 \right]$$

$$= \frac{1}{\sqrt{\bar{n}(1 + \bar{n})}} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^{1/2(\hat{x}-r)^2 + 1/2(\hat{p}-k)^2} \tag{3.16}$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators of the physical system. Therefore we obtain the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$ for the thermal filter state,

$$\hat{\Pi}(\Delta_r, \Delta_k) = \frac{\sinh(\theta/2)}{\pi} \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk \times \exp\left[-\frac{1}{2}\theta(\hat{x}-r)^2 - \frac{1}{2}\theta(\hat{p}-k)^2\right] = \frac{1}{2\pi\sqrt{\bar{n}(1+\bar{n})}} \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk \left(\frac{\bar{n}}{1+\bar{n}}\right)^{1/2(\hat{x}-r)^2 + 1/2(\hat{p}-k)^2} \tag{3.17}$$

It is easy to check that the probability $\mathrm{Tr}[\hat{\Pi}(\Delta_r, \Delta_k)\hat{\rho}]$ calculated from this equation is equal to that derived from (2.40).

The marginal operational probabilities $\mathcal{W}(\Delta_r)$ and $\mathcal{W}(\Delta_k)$ are obtained from equation (3.2),

$$\mathcal{W}_r(\Delta_r) = \mathcal{W}(\Delta_r, \mathbb{R}), \qquad \mathcal{W}_k(\Delta_k) = \mathcal{W}(\mathbb{R}, \Delta_k) \tag{3.18}$$

Thus it is found from (3.2), (3.8), and (3.18) that the marginal operational POVM $\hat{\Pi}_r(\Delta_r)$ and $\hat{\Pi}_k(\Delta_k)$ that satisfy $\mathcal{W}_r(\Delta_r) = \mathrm{TR}[\hat{\Pi}_r(\Delta_r)\hat{\rho}]$ and $\mathcal{W}_k(\Delta_k) = \mathrm{Tr}[\hat{\Pi}_k(\Delta_k)\hat{\rho}]$ for any statistical operator $\hat{\rho}$ of the physical system are given by

$$\hat{\Pi}_r(\Delta_r) \equiv \hat{\Pi}(\Delta_r, \mathbb{R}) = \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx\, |x\rangle f(x-r) \langle x| \tag{3.19}$$

$$\hat{\Pi}_k(\Delta_k) \equiv \hat{\Pi}(\mathbb{R}, \Delta_k) = \int_{k\in\Delta_k} dk \int_{-\infty}^{\infty} dp\, |p) g(p-k) (p| \tag{3.20}$$

where the filter functions $f(x)$ and $g(p)$ that represent the accuracy of the measurement apparatus are given by $f(x) = \langle x|\hat{\sigma}|x\rangle$ and $g(p) = (p|\sigma|p)$. To prove equation (3.19), we calculate the matrix element of the POVM $\hat{\Pi}_r(\Delta_r)$ with arbitrary position eigenstates $|x\rangle$ and $|y\rangle$. Thus we obtain from (3.8)

$$\langle x|\hat{\Pi}_r(\Delta_r)|y\rangle = \int_{r\in\Delta_r} dr \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \langle x|\hat{D}(r,k)\hat{\sigma}\hat{D}^\dagger(r,k)|y\rangle \tag{3.21}$$

Using the decomposition formula

$$e^{i(k\hat{x} - r\hat{p})} = e^{ik\hat{x}} e^{-ir\hat{p}} e^{-1/2ikr}$$

we calculate this equation

$$
\begin{aligned}
\langle x|\hat{\Pi}_r(\Delta_r)|y\rangle &= \int_{r\in\Delta_r} dr\,\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ik(x-y)}\langle x|e^{-ir\hat{p}}\hat{\sigma}e^{ir\hat{p}}|y\rangle \\
&= \delta(x-y)\int_{r\in\Delta_r} dr\,\langle x|e^{-ir\hat{p}}\,\hat{\sigma}e^{ir\hat{p}}|x\rangle \\
&= \delta(x-y)\int_{r\in\Delta_r} dr\,\langle x-r|\hat{\sigma}|x-r\rangle \\
&= \delta(x-y)\int_{r\in\Delta_r} dr\,f(x-r) \\
&= \langle x|\left(\int_{r\in\Delta_r} dr\int_{-\infty}^{\infty} dz\;|z\rangle f(z-r)\langle z|\right)|y\rangle
\end{aligned}
\tag{3.22}
$$

Therefore we have found that (3.19) is established. In the same way, we can prove (3.20) by calculating the matrix element of the POVM $\hat{\Pi}_k(\Delta_k)$ with arbitrary momentum eigenstates.

Recall that the sets of position and momentum eigenstates, $S_x = \{|x\rangle \mid x \in \mathbb{R}\}$ and $S_p = \{|p)|p \in \mathbb{R}\}$, are the complete orthonormal systems of the Hilbert space of the physical system. Then, using the eigenvalue equations, $f(\hat{x})|x\rangle = f(x)|x\rangle$ and $g(\hat{p})|p) = g(p)|p)$, we obtain from (3.19) and (3.20),

$$
\hat{\Pi}_r(\Delta_r) = \int_{r\in\Delta_r} dr\,f(\hat{x}-r) \tag{3.23}
$$

$$
\hat{\Pi}_k(\Delta_k) = \int_{k\in\Delta_k} dk\,g(\hat{p}-k) \tag{3.24}
$$

It is easily seen that the marginal operational POVM does not become a projection-valued measure and describes the unsharp quantum measurement of position or momentum of the physical system.

3.2. Naimark Extension and Relative-Coordinate States

We have found that the operational POVM and its marginal POVM for the operational phase-space measurement are not projection-valued measures. However, it is well known that by extending the Hilbert space, any POVM can be defined as a projection-valued measure on the extended Hilbert space (Helstrom, 1976; Holevo, 1982; Peres, 1993; Busch *et al.*, 1995), which is called the Naimark extensions of the POVM. Therefore, we consider the Naimark extensions of the operational POVM and investigate their physical properties.

For this purpose, we introduce an auxiliary Hilbert space $\mathcal{H}_a$ and define state vector $|u(r, k)\rangle\rangle$ in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$ (Ban, 1993a, b; 1996),

$$|u(r, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, |x + r\rangle \otimes |x\rangle_a e^{ikx} \tag{3.25}$$

Here we denote state vectors in the Hilbert spaces $\mathcal{H}$, $\mathcal{H}_a$, and $\mathcal{H} \otimes \mathcal{H}_a$ as $|\psi\rangle$, $|\psi\rangle_a$, and $|\psi\rangle\rangle$. Furthermore, $\hat{O}$ and $\hat{O}_a$ stand for operators defined on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_a$. It is easy to see that the set of the state vectors, $\{|u(r, k)\rangle\rangle | r, k \in \mathbb{R}\}$, is the complete orthonormal system in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$, which satisfies the relations,

$$\langle\langle u(r, k)|u(r', k')\rangle\rangle = \delta(r - r')\delta(k - k') \tag{3.26}$$

$$\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, |u(r, k)\rangle\rangle\langle\langle u(r, k)| = \hat{I} \otimes \hat{I}_a \tag{3.27}$$

where $\hat{I}$ and $\hat{I}_a$ are identity operators defined on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_a$. The state vector $|u(r, k)\rangle\rangle$ is the simultaneous eigenstate of operators $\hat{x} - \hat{x}_a$ and $\hat{p} + \hat{p}_a$ with eigenvalues r and k,

$$(\hat{x} - \hat{x}_a)|u(r, k)\rangle\rangle = r|u(r, k)\rangle\rangle \tag{3.28}$$

$$(\hat{p} + \hat{p}_a)|u(r, k)\rangle\rangle = k|u(r, k)\rangle\rangle \tag{3.29}$$

Thus we refer to the state vector $|u(r, k)\rangle\rangle$ as the relative-position state.

Let us introduce the bosonic annihilation and creation operators $\hat{b}$ and $\hat{b}^\dagger$ defined on the Hilbert space $\mathcal{H}_a$,

$$\hat{b} = \frac{\hat{x}_a + i\hat{p}_a}{\sqrt{2}}, \qquad b^\dagger = \frac{\hat{x}_a - i\hat{p}_a}{\sqrt{2}} \tag{3.30}$$

Using the annihilation and creation operators ($\hat{a}$, $\hat{a}^\dagger$, $\hat{b}$, $\hat{b}^\dagger$), we obtain the Fock-space representation of the relative-position state $|u(r, k)\rangle\rangle$ (Hongi-yi and Klauder, 1994; Hongi-yi and Xiong, 1995; Hongi-yi and Yue, 1996; Ban, 1996),

$$|u(r, k)\rangle\rangle = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}|\mu|^2 + \mu\hat{a}^\dagger - \mu^*\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger - \frac{1}{2}ikr\right)|0\rangle \otimes |0\rangle_a \tag{3.31}$$

where we set $\mu = (r + ik)/\sqrt{2}$, and $|0\rangle$ and $|0\rangle_a$ are the vacuum states, that is, $\hat{a}|0\rangle = 0$ and $\hat{b}|0\rangle_a = 0$.

In the same way, we can introduce the relative-momentum state $|v(r, k)\rangle\rangle$ in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}$,

$$|v(r,k)\rangle\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, |p + k) \otimes |p)_a e^{-ipr}$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}|\mu|^2 + \mu\hat{a}^\dagger + \mu^*\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger + \frac{1}{2} ikr\right)|0\rangle \otimes |0\rangle_a \quad (3.32)$$

It is easy to see that $|v(r, k)\rangle\rangle$ is the simultaneous eigenstate of operators $\hat{x} + \hat{x}_a$ and $\hat{p} - \hat{p}_a$ with eigenvalues r and k,

$$(\hat{x} + \hat{x}_a)|v(r,k)\rangle\rangle = r|v(r,k)\rangle\rangle \quad (3.33)$$

$$(\hat{p} - \hat{p}_a)|v(r,k)\rangle\rangle = k|v(r,k)\rangle\rangle \quad (3.34)$$

and that the set of the state vectors, $\{|v(r, k)\rangle\rangle \,|\, r, k \in \mathbb{R}\}$, becomes the complete orthonormal system in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$. Using the relative-position state $|u(r, k)\rangle\rangle$ and the relative-momentum state $|v(r, k)\rangle\rangle$, we can obtain the Naimark extensions of the operational POVM and the marginal POVM.

We now construct a projection-valued measure defined on the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$ in terms of the relative-position state $|u(r, k)\rangle\rangle$ given by (3.25) or (3.31), which gives the same quantum probability as that obtained by the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$. Consider the operator $\hat{\mathcal{X}}(\Delta_r, \Delta_k)$ defined on the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$,

$$\hat{\mathcal{X}}(\Delta_r, \Delta_k) = \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk\, |u(r,k)\rangle\rangle\, \langle\langle u(r,k)| \quad (3.35)$$

which satisfies the relations

$$\hat{\mathcal{X}}(\Delta_r, \Delta_k) \geq 0, \qquad \hat{\mathcal{X}}(\mathbb{R}, \mathbb{R}) = \hat{I} \otimes \hat{I}_a, \qquad \hat{\mathcal{X}}(\Delta_r, \emptyset) = \hat{\mathcal{X}}(\emptyset, \Delta_k) = 0 \quad (3.36)$$

$$\hat{\mathcal{X}}(\Delta_r^{(1)} \cup \Delta_r^{(2)}, \Delta_k) = \hat{\mathcal{X}}(\Delta_r^{(1)}, \Delta_k) + \hat{\mathcal{X}}(\Delta_r^{(2)}, \Delta_k) \quad (3.37)$$

$$\hat{\mathcal{X}}(\Delta_r, \Delta_k^{(1)} \cup \Delta_k^{(2)}) = \hat{\mathcal{X}}(\Delta_r, \Delta_k^{(1)}) + \hat{\mathcal{X}}(\Delta_r, \Delta_k^{(2)}) \quad (3.38)$$

$$\hat{\mathcal{X}}(\Delta_r, \Delta_k)\hat{\mathcal{X}}(\Delta_r', \Delta_k') = \hat{\mathcal{X}}(\Delta_r \cap \Delta_r', \Delta_k \cap \Delta_k') \quad (3.39)$$

where $\Delta_r^{(1)}$ and $\Delta_r^{(2)}$ ($\Delta_k^{(1)}$ and $\Delta_k^{(2)}$) are disjoined subsets of $\mathbb{R}$. In particular, setting $\Delta_r = \Delta_k'$ and $\Delta_k = \Delta_k'$ in (3.39), we obtain

$$[\hat{\mathcal{X}}(\Delta_r, \Delta_k)]^2 = \hat{\mathcal{X}}(\Delta_r, \Delta_k) \quad (3.40)$$

Therefore it is found from equations (3.36)–(3.40) that the operator $\hat{\mathcal{X}}(\Delta_r, \Delta_k)$ is a projection-valued measure defined on the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$.

To proceed further, let us introduce a statistical operator $\hat{\rho}_a$ defined on the auxiliary Hilbert space $\mathscr{H}_a$,

$$\hat{\rho}_a = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m\rangle_a \, [\langle n|\hat{\sigma}|m\rangle]_a\langle n| \tag{3.41}$$

where $\hat{\sigma}$ is the quantum-filter state defined on the Hilbert space $\mathscr{H}$. It is seen that the statistical operator $\hat{\rho}_a$ satisfies the relations

$$_a\langle m|\hat{\rho}_a|n\rangle_a = \langle n|\hat{\sigma}|m\rangle, \qquad _a\langle x|\hat{\rho}_a|y\rangle_a = \langle y|\hat{\sigma}|x\rangle, \qquad _a(u|\hat{\rho}_a|v)_a = (-v|\hat{\sigma}| - u) \tag{3.42}$$

where we have used the relations

$$\langle n|x\rangle = \frac{1}{\sqrt{\pi^{1/2}2^n n!}} e^{-1/2x^2} H_n(x) = \langle x|n\rangle \tag{3.43}$$

$$\langle n|p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{ipx} \langle n|x\rangle = (-p|n\rangle \tag{3.44}$$

Here $H_n(x)$ is the Hermite polynomial of order n. Using (3.8), (3.25), (3.35), and (3.41), we can calculate as follows:

$$\begin{aligned}
\mathrm{Tr}\mathrm{Tr}_a[\hat{\mathscr{X}}(\Delta_r, \Delta_k)\hat{\rho} \otimes \hat{\rho}_a] &= \int_{r\in\Delta_r} dr \int_{r\in\Delta_k} dk \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\
&\quad \times e^{-ik(x-y)} \langle x + r|\hat{\rho}|y + r\rangle \, _a\langle x|\hat{\rho}_a|y\rangle_a \\
&= \int_{r\in\Delta_r} dr \int_{r\in\Delta_k} dk \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\
&\quad \times e^{-ik(x-y)} \langle y|\hat{\sigma}|x\rangle\langle x + r|\hat{\rho}|y + r\rangle \\
&= \int_{r\in\Delta_r} dr \int_{r\in\Delta_k} dk \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\
&\quad \times \langle y|e^{ik\hat{x}}\hat{\sigma}e^{-ik\hat{x}}|x\rangle\langle x|e^{ir\hat{p}}\hat{\rho}e^{-ir\hat{p}}|y\rangle \\
&= \int_{r\in\Delta_r} dr \int_{r\in\Delta_k} dk \frac{1}{2\pi} \mathrm{Tr}[e^{-ir\hat{p}}e^{ik\hat{x}}\hat{\sigma}e^{-ik\hat{x}}e^{ir\hat{p}}\hat{\rho}] \\
&= \int_{r\in\Delta_r} dr \int_{r\in\Delta_k} dk \frac{1}{2\pi} \mathrm{Tr}[\hat{D}(r, k)\hat{\sigma}\hat{D}^{\dagger}(r, k)\hat{\sigma}] \\
&= \mathrm{Tr}[\hat{\Pi}(\Delta_r, \Delta_k)\hat{\rho}]
\end{aligned} \tag{3.45}$$

Therefore we find from equation (3.6) that the following relation is established:

$$\mathcal{W}(\Delta_r, \Delta_k) = \mathrm{Tr}[\hat{\Pi}(\Delta_r, \Delta_k)\hat{\rho}] = \mathrm{Tr}\mathrm{Tr}_a[\hat{\mathcal{X}}(\Delta_r, \Delta_k)\hat{\rho} \otimes \hat{\rho}_a] \tag{3.46}$$

This result indicates that the projection-valued measure $\hat{\mathcal{X}}(\Delta_r, \Delta_k)$ and the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$ give the same quantum probability $\mathcal{W}(\Delta_r, \Delta_k)$. Thus the projection-valued measure $\hat{\mathcal{X}}(\Delta_r, \Delta_k)$ is the Naimark extension of the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$. The quantum measurement described by the projection-valued measure $\hat{\mathcal{X}}(\Delta_r, \Delta_k)$ is the simultaneous measurement of the position-difference and the momentum-sum, $\hat{x} - \hat{x}_a$ and $\hat{p} + \hat{p}_a$, in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$. Such a quantum measurement can be implemented in some quantum optical systems (Lai and Haus, 1989; Ban, 1996).

There is another Naimark extension $\hat{\mathcal{Y}}(\Delta_r, \Delta_k)$ of the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$ which is constructed in terms of the relative momentum state $|v(r, k)\rangle\rangle$ given by (3.32),

$$\hat{\mathcal{Y}}(\Delta_r, \Delta_k) = \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk\, |v(r, k)\rangle\rangle\, \langle\langle v(r, k)| \tag{3.47}$$

which satisfies the relations obtained by replacing $\hat{\mathcal{X}}$ with $\hat{\mathcal{Y}}$ in (3.36)–(3.40). Here we introduce a statistical operator $\hat{\rho}_a'$ defined on the auxiliary Hilbert $\mathcal{H}_a$,

$$\rho_a' = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m\rangle_a[(-1)^{m+n}\langle n|\hat{\sigma}|m\rangle]_a\langle n| \tag{3.48}$$

which satisfies the relations

$$_a\langle x|\hat{\rho}_a'|y\rangle_a = \langle -y|\hat{\sigma}| - x\rangle, \qquad _a(u|\hat{\rho}_a'|v)_a = (v|\hat{\sigma}|u) \tag{3.49}$$

Therefore, using the same method of deriving (3.46), we can obtain the expression for the operational phase-space probability,

$$\mathcal{W}(\Delta_r, \Delta_k) = \mathrm{Tr}[\hat{\Pi}(\Delta_r, \Delta_k)\hat{\rho}] = \mathrm{Tr}\mathrm{Tr}_a[\hat{\mathcal{Y}}(\Delta_r, \Delta_k)\hat{\rho} \otimes \hat{\rho}_a'] \tag{3.50}$$

The quantum measurement described by the projection-valued measure $\hat{\mathcal{Y}}(\Delta_r, \Delta_k)$ is the simultaneous measurement of the position-sum and the momentum-difference, $\hat{x} + \hat{x}_a$ and $\hat{p} - \hat{p}_a$, in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$. As seen from the above results, the Naimark extension of the operational POVM is not unique. This means that there are many standard quantum measurements in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$ that correspond to the operational phase-space measurement in the original Hilbert space $\mathcal{H}$. It should be noted that the statistical operators $\hat{\rho}_a$ and $\hat{\rho}_a'$ are related by

$$\rho_a' = \hat{\mathcal{P}}_a\hat{\rho}_a\hat{\mathcal{P}}_a, \qquad \hat{\rho}_a = \hat{\mathcal{P}}_a\hat{\rho}_a'\hat{\mathcal{P}}_a \tag{3.51}$$

where $\hat{\mathcal{P}}_a$ is the parity operator defined on the Hilbert space $\mathcal{H}_a$ (Royer, 1977),

$$\hat{\mathcal{P}}_a = \int_{-\infty}^{\infty} dx\ |-x\rangle_a\ {}_a\langle x| = \int_{-\infty}^{\infty} dp\ |-p)_a\ {}_a(p| \tag{3.52}$$

which satisfies the relations $\hat{\mathcal{P}}_a^\dagger = \hat{\mathcal{P}}_a$ and $\hat{\mathcal{P}}_a^2 = \hat{I}_a$.

Next we consider the Naimark extension of the marginal operational POVM given by (3.19) and (3.20). Using the projection-valued measures $\hat{\mathcal{X}}(\Delta_r, \Delta_k)$ and $\hat{\mathcal{Y}}(\Delta_r, \Delta_k)$, we find from (3.25) and (3.32) that their marginal projection-valued measures defined on the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$ are given by

$$\hat{\mathcal{X}}_r(\Delta_r) = \hat{\mathcal{X}}(\Delta_r, \mathbb{R}) = \int_{r \in \Delta_r} dr \int_{-\infty}^{\infty} dx\ |x + r\rangle\langle x + r| \otimes |x\rangle_a\ {}_a\langle x| \tag{3.53}$$

$$\hat{\mathcal{X}}_k(\Delta_k) = \hat{\mathcal{X}}(\mathbb{R}, \Delta_k) = \int_{k \in \Delta_k} dk \int_{-\infty}^{\infty} dp\ |k - p)(k - p| \otimes |p)_a\ {}_a(p| \tag{3.54}$$

$$\hat{\mathcal{Y}}_r(\Delta_r) = \hat{\mathcal{Y}}(\Delta_r, \mathbb{R}) = \int_{r \in \Delta_r} dr \int_{-\infty}^{\infty} dx\ |r - x\rangle\langle r - x| \otimes |x\rangle_a\ {}_a\langle x| \tag{3.55}$$

$$\hat{\mathcal{Y}}_k(\Delta_k) = \hat{\mathcal{Y}}(\mathbb{R}, \Delta_k) = \int_{k \in \Delta_k} dk \int_{-\infty}^{\infty} dp\ |p + k)(p + k| \otimes |p)_a\ {}_a(p| \tag{3.56}$$

It is easy to see from (3.46) and (3.50) that the marginal phase-space probability distributions $\mathcal{W}_r(\Delta_r)$ and $\mathcal{W}_k(\Delta_k)$ are expressed as

$$\begin{aligned} \mathcal{W}_r(\Delta_r) &= \mathrm{Tr}[\hat{\Pi}_r(\Delta_r)\hat{\rho}] \\ &= \mathrm{Tr}\mathrm{Tr}_a[\hat{\mathcal{X}}_r(\Delta_r)\hat{\rho} \otimes \hat{\rho}_a] = \mathrm{Tr}\mathrm{Tr}_a[\hat{\mathcal{Y}}_r(\Delta_r)\hat{\rho} \otimes \hat{\rho}_a'] \end{aligned} \tag{3.57}$$

$$\begin{aligned} \mathcal{W}_k(\Delta_k) &= \mathrm{Tr}[\hat{\Pi}_k(\Delta_k)\hat{\rho}] \\ &= \mathrm{Tr}\mathrm{Tr}_a[\hat{\mathcal{X}}_k(\Delta_k)\hat{\rho} \otimes \hat{\rho}_a] = \mathrm{Tr}\mathrm{Tr}_a[\hat{\mathcal{Y}}_k(\Delta_k)\hat{\rho} \otimes \rho_a'] \end{aligned} \tag{3.58}$$

where the statistical operators $\hat{\rho}_a$ and $\hat{\rho}_a'$ are given, respectively, by (3.41) and (3.48) and we have used relations (3.42) and (3.49). Therefore we have found that the projection-valued measures $\hat{\mathcal{X}}_r(\Delta_r)$ $[\hat{\mathcal{Y}}_r(\Delta_r)]$ and $\hat{\mathcal{X}}_k(\Delta_k)$ $[\hat{\mathcal{Y}}_k(\Delta_k)]$ are the Naimark extensions of the marginal operational POVM $\hat{\Pi}_r(\Delta_r)$ $[\hat{\Pi}_k(\Delta_k)]$.

3.3. Measurable Quantity and Fuzzy Observable

We finally consider the observable quantities measured in the operational phase-space measurement. Since we have obtained the POVM for the operational phase-space measurement, it is easy to obtain the observable quantities. It should be noted here that the operational POVM is not a projection-valued measure and so the observable quantity does not become a Hermitian operator. Such a physical quantity is called a semiobservable, unsharp observable, or fuzzy observable (Prugovečki, 1973, 1974, 1975, 1976a, b, 1977; Twareque Ali and Emch, 1974; Twareque Ali and Doebner, 1976; Twareque Ali and Prugovečki, 1977a, b; Morato, 1977).

Recall that the operational position measurement on the physical system is described by the operational POVM $\hat{\Pi}_r(\Delta_r)$ given by (3.19) or (3.23). Thus the measured quantity, which is an analytic function $\mathcal{F}(x)$ of position, is expressed as

$$\widehat{\mathcal{F}(x)} = \int_{-\infty}^{\infty} \mathcal{F}(x)\, \hat{\Pi}_r(dx) \tag{3.59}$$

where $\hat{\Pi}_r(dx) = \hat{\Pi}_r(\Delta_r)|_{\Delta_r \to (x, x+dx)}$. If the operational POVM were a projection-valued measure, this equation would have been the spectral decomposition of the Hermitian operator $\mathcal{F}(\hat{x})$. For example, when $\mathcal{F}(x) = x^n$, substituting (3.19) or (3.23) into (3.59), the measured quantity (fuzzy observable) $\widehat{x^n}$ becomes

$$\begin{aligned} \widehat{x^n} &= \int_{-\infty}^{\infty} dr\, r^n f(\hat{x} - r) \\ &= \int_{-\infty}^{\infty} dr\, (\hat{x} - r)^n f(r) \\ &= \sum_{m=0}^{n} \binom{n}{m} (-1)^m \hat{x}^{n-m} \int_{-\infty}^{\infty} dr\, r^m f(r) \\ &= \sum_{m=0}^{n} \binom{n}{m} (-1)^m \hat{x}^{n-m} \mathrm{Tr}[\hat{x}^m \hat{\sigma}] \end{aligned} \tag{3.60}$$

where $\binom{n}{m}$ is the binomial coefficient. It is easily seen from this equation that $\widehat{x^n} \neq (\hat{x})^n$, which is characteristic of the unsharp or fuzzy observable. When $\hat{\mathcal{F}}(x)$ is an analytic function which is expanded as $\hat{\mathcal{F}}(x) = \Sigma_n \mathcal{F}_n x^n$, we obtain the fuzzy observable $\widehat{\mathcal{F}(x)}$ measured in the operational position measurement,

$$\widehat{\mathcal{F}(x)} = \sum_n \mathcal{F}_n \sum_{m=0}^{n} \binom{n}{m} (-1)^m \hat{x}^{n-m} \mathrm{Tr}[\hat{x}^m \hat{\sigma}] \tag{3.61}$$

In the same way, using (3.20) or (3.24), we can obtain the fuzzy momentum observable $\widehat{\mathcal{G}(k)}$ in the following form:

$$\begin{aligned} \widehat{\mathcal{G}(p)} &= \int_{-\infty}^{\infty} \mathcal{G}(k)\, \hat{\Pi}_k(dk) \\ &= \sum_n \mathcal{G}_n \sum_{m=0}^{n} \binom{n}{m} (-1)^m \hat{p}^{n-m} \mathrm{Tr}[\hat{p}^m \hat{\sigma}] \end{aligned} \tag{3.62}$$

where $\hat{\Pi}_k(dp) = \hat{\Pi}_k(\Delta_k)|_{\Delta_k \to (p,p+dp)}$ and we have assumed that the function $\mathcal{G}(k)$ is expanded as $\mathcal{G}(k) = \Sigma_n \mathcal{G}_n k^n$. It is easy to see from (3.61) and (3.62) that

$$\widehat{\mathcal{F}(x)} \neq \mathcal{F}(\hat{x}), \qquad \widehat{\mathcal{G}(p)} \neq \mathcal{G}(\hat{p}) \tag{3.63}$$

In particular, when we set $\mathcal{F}(x) = \exp(i\mu x)$ and $\mathcal{G}(p) = \exp(i\mu p)$, we obtain

$$\widehat{\exp(i\mu x)} = \exp(i\mu\hat{x}) \operatorname{Tr}[\exp(-i\mu\hat{x})\hat{\sigma}] \tag{3.64}$$

$$\widehat{\exp(i\mu p)} = \exp(i\mu\hat{p}) \operatorname{Tr}[\exp(-i\mu\hat{p})\hat{\sigma}] \tag{3.65}$$

which yield the characteristic functions for the operational position and momentum measurements.

Before closing this section, we consider the squeezed-vacuum state as the quantum-filter state, namely, $\hat{\sigma} = |\gamma\rangle\langle\gamma|$, where the quantum state $|\gamma\rangle$ is given by (2.32) and the squeezing parameter γ is real. In this case, the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$ is given by

$$\hat{\Pi}(\Delta_r, \Delta_k) = \frac{1}{\pi} \int_{\mu \in \tilde{\Delta}_r \times \tilde{\Delta}_k} d^2\mu \, |\mu, \gamma\rangle\langle\mu, \gamma| \tag{3.66}$$

where $|\mu, \gamma\rangle = \hat{D}(\mu)\hat{S}(\gamma)|0\rangle$ is the coherent-squeezed state of the physical system (Yuen, 1976; Walls and Milburn, 1994). Then we obtain the marginal operational POVM,

$$\hat{\Pi}_r(\Delta_r) = \frac{1}{e^{\gamma}\sqrt{\pi}} \int_{r \in \Delta_r} dr \exp\left[-\left(\frac{\hat{x} - r}{e^{\gamma}}\right)^2\right] \tag{3.67}$$

$$\hat{\Pi}_k(\Delta_k) = \frac{1}{e^{-\gamma}\sqrt{\pi}} \int_{k \in \Delta_k} dk \exp\left[-\left(\frac{\hat{p} - k}{e^{-\gamma}}\right)^2\right] \tag{3.68}$$

by means of which the fuzzy observables of the nth power of position and momentum are calculated to be

$$\widehat{x^n} = \sum_{j=0}^{[n/2]} \binom{n}{2j} \frac{\Gamma(j + \frac{1}{2})}{\Gamma(\frac{1}{2})} \exp(2j\gamma)\hat{x}^{n-2j} \tag{3.69}$$

$$\widehat{p^n} = \sum_{j=0}^{[n/2]} \binom{n}{2j} \frac{\Gamma(j + \frac{1}{2})}{\Gamma(\frac{1}{2})} \exp(-2j\gamma)\hat{p}^{n-2j} \tag{3.70}$$

where $\Gamma(x)$ is the gamma function (Abramowitz and Stegun, 1970) and $[x]$ is the maximum value of the integer less than or equal to x. In particular, we have

$$\widehat{x} = \hat{x}, \qquad \widehat{x^2} = \hat{x}^2 + \tfrac{1}{2}e^{2\gamma} \tag{3.71}$$

$$\widehat{p} = \hat{p}, \qquad \widehat{p^2} = \hat{p}^2 + \tfrac{1}{2}e^{-2\gamma} \tag{3.72}$$

The observable quantity measured in the operational phase-space measurement includes the fluctuation caused by the measurement apparatus in addition to the intrinsic fluctuation appearing in the quantum state of the physical system. In particular, when $-\gamma >> 1$ ($\gamma >> 1$), we obtain $\widehat{x^2} = \hat{x}^2$ ($\widehat{p^2} = \hat{p}^2$). In this case, the additional fluctuation caused by the measurement apparatus does not come in the position (momentum) measurement, though the result of the momentum (position) measurement becomes completely uncertain.

4. STATE CHANGE IN OPERATIONAL PHASE-SPACE MEASUREMENT

4.1. State Change in Position Measurement

When we perform a quantum measurement on a physical system, we can obtain some information about the quantum state of the measured physical system. At the same time, the quantum state of the physical system is changed into another quantum state by the effect of the quantum measurement from which we obtain the information. Therefore, in this section, we investigate the state change of the physical system caused by the operational phase-space measurement considered in Sections 2 and 3. The state change of the physical system is mathematically described by an operation or a completely positive instrument in the most general way (Davies, 1976; Helstrom, 1976; Holevo, 1982; Kraus, 1983; Busch *et al.*, 1995; Ozawa, 1984, 1993). So we will obtain the operation that describes the state change caused by the operational phase-space measurement. An arbitrary operation is expressed as a superoperator which transforms an operator into another operator (Fano, 1957; Crawford, 1958; Prigogine *et al.*, 1973; Schmutz, 1978; Umezawa, 1993). It should be noted that superoperators are equivalent to thermofields (Umezawa *et al.*, 1982; Umezawa, 1993).

Let $\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)$ be the operation that describes the state change of the physical system when the outcomes r and k of the operational phase-space measurement belong to the ranges Δ_r and Δ_k. It should be noted that in this paper, a symbol with double caret such as $\hat{\hat{O}}$ stands for a superoperator and a symbol with single caret such as $\hat{O}$ represents an operator. Then, using the operation $\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)$, we express the state change of the physical system as

$$\hat{\rho} \to \frac{\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{\rho}}{\mathrm{Tr}[\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{\rho}]} \tag{4.1}$$

where we have assumed that $\mathrm{Tr}[\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{\rho}] \neq 0$. The probability that such a state change occurs in the operational phase-space measurement is given by $\mathrm{Tr}[\hat{\hat{\mathscr{L}}}(\Delta_r, \Delta_k)\hat{\rho}]$. Thus the operation $\hat{\hat{\mathscr{L}}}(\Delta_r, \Delta_k)$ is related to the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$ by

$$\mathscr{W}(\Delta_r, \Delta_k) = \mathrm{Tr}[\hat{\Pi}(\Delta_r, \Delta_k)\hat{\rho}] = \mathrm{Tr}[\hat{\hat{\mathscr{L}}}(\Delta_r, \Delta_k)\hat{\rho}] \tag{4.2}$$

where $\mathscr{W}(\Delta_r, \Delta_k)$ and $\hat{\Pi}(\Delta_r, \Delta_k)$ are given by (2.4) and (3.8). To obtain the operation $\hat{\hat{\mathscr{L}}}(\Delta_r, \Delta_k)$ which satisfies the relations (4.1) and (4.2), we have to assume the interaction Hamiltonian between the measured physical system and the measurement apparatus. We first investigate the operations for the operational position and momentum measurement and then we consider the simultaneous measurement of position and momentum.

We now consider the standard model of the position measurement on a physical system, where the interaction Hamiltonian between the measured physical system and the measurement apparatus is given by

$$\hat{H}_{\mathrm{int}} = g\hat{x} \otimes \hat{p}_a \tag{4.3}$$

where the parameter g represents the strength of the interaction and $\hat{p}_a$ is the canonical momentum operator of the measurement apparatus. Any quantity concerned with the measurement apparatus is expressed with a subscript a, such as $\hat{O}$ and $|\psi\rangle_a$. To investigate the state change induced by the measurement process, suppose that the physical system is in the quantum state $\hat{\rho}$ and the measurement apparatus is prepared in the quantum state $\hat{\rho}_a$ before the interaction. Furthermore, we assume that the interaction is turned on at $t = 0$ and the duration of the interaction is $\rho = 1/g$, for the sake of simplicity. Then the composite quantum state $\hat{W}$ of the physical system and the measurement apparatus after the interaction becomes

$$\hat{W} = \hat{V}(\hat{\rho} \otimes \hat{\rho}_a)\hat{V}^\dagger \tag{4.4}$$

with

$$\hat{V} = \exp(-i\hat{x} \otimes \hat{p}_a) \tag{4.5}$$

It is important to note that what we know about the physical system is the outcome exhibited by the measurement apparatus, while we cannot directly get the value of the physical quantity of the system. Thus the situation in which the measurement apparatus shows the position value in the range Δ_r is described by the projection-valued measure of the measurement apparatus,

$$\hat{\mathscr{X}}_a(\Delta_r) = \int_{r\in\Delta_r} dr\, |r\rangle_a\, {}_a\langle r| \tag{4.6}$$

where $|r\rangle_a$ is the eigenstate of the position operator $\hat{x}_a$ of the measurement apparatus with eigenvalue r. When we obtain the measurement outcome $r \in$

Δ_r, the nonnormalized statistical operator $\hat{W}_r(\Delta_r)$ of the quantum state of the physical system after the measurement is calculated from (4.4)–(4.6),

$$\begin{aligned}
\hat{W}_r(\Delta_r) &= \mathrm{Tr}_a[(\hat{I} \otimes \hat{\mathscr{X}}_a(\Delta_r))\hat{W}] \\
&= \int_{r\in\Delta_r} dr \; {}_a\langle r|\hat{V}(\hat{\rho} \otimes \hat{\rho}_a)\hat{V}^\dagger|r\rangle_a \\
&= \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; |x\rangle\langle x|\hat{\rho}|y\rangle\langle y| \\
&\quad \times (e^{-x\, d/dr}\, {}_a\langle r|)\hat{\rho}_a(e^{-y\, d/dr}|r\rangle_a) \\
&= \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; |x\rangle\langle x|\hat{\rho}|y\rangle\langle y|_a\langle r - x|\hat{\rho}_a|r - y\rangle_a
\end{aligned} \tag{4.7}$$

where we have ignored the unimportant free time-evolutions of the physical system and the measurement apparatus, for the sake of simplicity.

To proceed further, we introduce the spectral decomposition of the statistical operator $\hat{\rho}_a$ of the measurement apparatus,

$$\hat{\rho}_a = \sum_{j\in\mathscr{S}} p_j|\psi_j\rangle_a\, {}_a\langle\psi_j| \tag{4.8}$$

where $\mathscr{S}$ stands for the spectral set of the statistical operator $\hat{\rho}_a$ and the eigenvalue p_j satisfies the relations

$$p_j \geq 0, \qquad \sum_{j\in\mathscr{S}} p_j = 1 \tag{4.9}$$

Substituting (4.8) into (4.7), we find that the nonnormalized statistical operator $\hat{W}_r(\Delta_r)$ becomes

$$\begin{aligned}
\hat{W}_r(\Delta_r) &= \int_{r\in\Delta_r} dr \sum_{j\in\mathscr{S}} p_j \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\
&\quad \times |x\rangle\langle x|\hat{\rho}|y\rangle\langle y|\psi_j(r - x)\psi_j^*(r - y) \\
&= \int_{r\in\Delta_r} dr \sum_{j\in\mathscr{S}} p_j \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\
&\quad \times |x\rangle\langle x|\psi_j(r - \hat{x})\hat{\rho}\psi_j^\dagger(r - \hat{x})|y\rangle\langle y| \\
&= \int_{r\in\Delta_r} dr \sum_{j\in\mathscr{S}} p_j\psi_j(r - \hat{x})\hat{\rho}\psi_j^\dagger(r - \hat{x})
\end{aligned} \tag{4.10}$$

where the wave function $\psi_j(x)$ is given by $\psi_j(x) = {}_a\langle x|\psi_j\rangle_a$ and we have used the eigenvalue equation of the position operator $\hat{x}$ of the physical system such as

$$\psi_j(r - \hat{x})|y\rangle = \psi_j(r - y)|y\rangle \tag{4.11}$$

Therefore from equations (4.7) and (4.10), we obtain the operation $\hat{\hat{\mathcal{L}}}_r(\Delta_r)$ that describes the state change of the physical system caused by the position measurement,

$$\begin{aligned}\hat{\hat{\mathcal{L}}}_r(\Delta_r)\hat{O} &= \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathcal{K}_p(r - x, r - y)|x\rangle\langle x|\hat{O}|y\rangle\langle y| \\ &= \int_{r\in\Delta_r} dr \sum_{j\in\mathcal{S}} p_j\psi_j(r - \hat{x})\hat{O}\psi_j^\dagger(r - \hat{x})\end{aligned} \tag{4.12}$$

where the integral kernel $\mathcal{K}_p(x, y)$ is given by $\mathcal{K}_p(x, y) = {}_a\langle x|\hat{\rho}_a|y\rangle_a$ and $\hat{O}$ stands for an arbitrary operatory defined on the Hilbert space of the physical system. When we introduce superoperators $\hat{\hat{u}}_+$ and $\hat{\hat{u}}_-$ by $\hat{\hat{u}}_+\hat{A} = \hat{u}A$ and $\hat{\hat{u}}_-A = \hat{A}\hat{u}^\dagger$ for any operator $\hat{A}$, it is easy to see that the operation $\hat{\hat{\mathcal{L}}}_r(\Delta_r)$ is expressed as

$$\hat{\hat{\mathcal{L}}}_r(\Delta_r) = \int_{r\in\Delta_r} dr\, \mathcal{K}_p(r - \hat{\hat{x}}_+, r - \hat{\hat{x}}_-)$$

Our next task is to show that the operation $\hat{\hat{\mathcal{L}}}_r(\Delta_r)$ given by (4.12) describes the state change of the operational phase-space measurement of position. To this end, we have to show that the relation (4.2) is satisfied by the operation $\hat{\hat{\mathcal{L}}}_r(\Delta_r)$, where the operational POVM $\hat{\Pi}_r(\Delta_r)$ is given by (3.19) or (3.23). If the relation is valid for the operation $\hat{\hat{\mathcal{L}}}_r(\Delta_r)$, the POVM $\hat{\Pi}_r(\Delta_r)$ in (4.2) is calculated as

$$\begin{aligned}\hat{\Pi}_r(\Delta_r) &= \int_{r\in\Delta_r} dr \sum_{j\in\mathcal{S}} p_j\psi_j^\dagger(r - \hat{x})\psi_j(r - \hat{x}) \\ &= \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx \sum_{j\in\mathcal{S}} p_j|\psi_j(r - x)|^2|x\rangle\langle x| \\ &= \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx\, {}_a\langle r - x|\hat{\rho}_a|r - x\rangle_a|x\rangle\langle x|\end{aligned} \tag{4.13}$$

Here let us define the quantum-filter state $\hat{\sigma}$ of the physical system by

$$\langle x|\hat{\sigma}|y\rangle = \sum_{j\in\mathcal{S}} p_j\psi_j(-y)\psi_j^*(-x) \tag{4.14}$$

Using this quantum-filter state, the POVM $\hat{\Pi}_r(\Delta_r)$ becomes

$$\begin{aligned}\hat{\Pi}_r(\Delta_r) &= \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx\, \langle x - r|\hat{\sigma}|x - r\rangle |x\rangle\langle x| \\ &= \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx\, f(x - r)|x\rangle\langle x| \\ &= \int_{r\in\Delta_r} dr\, f(\hat{x} - r) \end{aligned} \tag{4.15}$$

where $f(x) = \langle x|\hat{\sigma}|x\rangle$. Comparing this result with (3.19) or (3.23), we find that $\hat{\Pi}_r(\Delta_r)$ is the operational POVM of the phase-space position measurement. Thus the operation $\hat{\hat{\mathcal{L}}}_r(\Delta_r)$ given by (4.12) describes the state change of the physical system caused by the operational phase-space measurement of position.

Therefore, when we obtain the measurement outcome $r \in \Delta_r$, the statistical operator $\hat{\rho}_r(\Delta_r)$ of the quantum state of the physical system after the measurement is given by

$$\hat{\rho}_r(\Delta_r) = \frac{\hat{\hat{\mathcal{L}}}_r(\Delta_r)\hat{\rho}}{\mathrm{Tr}[\hat{\hat{\mathcal{L}}}_r(\Delta_r)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}_r(\Delta_r)\hat{\rho}}{\mathrm{Tr}[\hat{\Pi}_r(\Delta_r)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}_r(\Delta_r)\hat{\rho}}{\mathcal{W}_r(\Delta_r)} \tag{4.16}$$

This result will be used to investigate the information about the physical system extracted from the operational phase-space measurement in Section 5.

4.2. State Change in Momentum Measurement

We next consider the state change of the physical system caused by the momentum measurement. In this case, the interaction between the physical system and the measurement apparatus is given by the interaction Hamiltonian,

$$\hat{H}_{\mathrm{int}} = g\hat{p} \otimes \hat{p}_a \tag{4.17}$$

Remember that we denote the eigenstate of the position operator as $|u\rangle$ and the eigenstate of the momentum operator as $|u)$. Thus, for example, we have $\hat{x}|k\rangle = k|k\rangle$ and $\hat{p}|r) = r|r)$. When the quantum states of the physical system and the measurement apparatus before the interaction are given by the statistical operators $\hat{\rho}$ and $\hat{\rho}_a$, their composite quantum state $\hat{W}$ after the interaction becomes

$$\hat{W} = \hat{U}(\hat{\rho} \otimes \hat{\rho}_a)\hat{U}^\dagger \tag{4.18}$$

with

$$\hat{U} = \exp(-i\hat{p} \otimes \hat{p}_a) \tag{4.19}$$

where we set the initial time $t = 0$ and the interaction time $\tau = 1/g$.

Suppose that we read the value of the position variable of the measurement apparatus to get the information about the momentum of the physical system. The measurement apparatus that shows the position value in the range Δ_k is described by the projection-valued measure, $\hat{\mathscr{X}}_a(\Delta_k) = \int_{x\in\Delta_k} dx\, |x\rangle_a\, {}_a\langle x|$. Thus, when we obtain the measurement outcome $k \in \Delta_k$, the nonnormalized statistical operator $\hat{W}_k(\Delta_k)$ of the quantum state of the physical system after the measurement is calculated as

$$\begin{aligned}
\hat{W}_k(\Delta_k) &= \mathrm{Tr}_a[(\hat{I} \otimes \hat{\mathscr{X}}_a(\Delta_k))\hat{W}] \\
&= \int_{k\in\Delta_k} dk\, {}_a\langle k|\hat{U}(\hat{\rho} \otimes \hat{\rho}_a)\hat{U}^\dagger|k\rangle_a \\
&= \int_{k\in\Delta_k} dk \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, |u)(u|\hat{\rho}|v)(v|(e_a^{-u\, d/dk}\langle k|)\hat{\rho}_a(e^{-v\, d/dk}|k\rangle_a) \\
&= \int_{k\in\Delta_k} dk \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, |u)(u|\hat{\rho}|v)(v|_a\langle k-u|\hat{\rho}_a|k-v\rangle_a
\end{aligned} \tag{4.20}$$

Furthermore, assuming the spectral decomposition of the statistical operator $\hat{\rho}_a$ given by (4.8), we obtain the expression

$$\hat{W}_k(\Delta_k) = \int_{k\in\Delta_k} dk \sum_{j\in\mathscr{S}} p_j\psi_j(k-\hat{p})\hat{\rho}\psi_j^\dagger(k-\hat{p}) \tag{4.21}$$

where $\psi_j(k) = {}_a\langle k|\psi_j\rangle_a$ and we have used the eigenvalue equations of the momentum operator such as $\psi_j(k-\hat{p})|u) = \psi_j(k-u)|u)$. Therefore we obtain the operation $\hat{\hat{\mathscr{L}}}_k(\Delta_k)$ that describes the state change of the physical system caused by the momentum measurement,

$$\begin{aligned}
\hat{\hat{\mathscr{L}}}_k(\Delta_k)\hat{O} &= \int_{k\in\Delta_k} dk \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \mathscr{K}_m(k-u, k-v)|u)(u|\hat{O}|v)(u| \\
&= \int_{k\in\Delta_k} dk \sum_{j\in\mathscr{S}} p_j\psi_j(k-\hat{p})\hat{O}\psi_j^\dagger(k-\hat{p})
\end{aligned} \tag{4.22}$$

where we have defined the integral kernel $\mathscr{K}_m(u, v) = {}_a\langle u|\hat{\rho}_a|v\rangle_a$ and $\hat{O}$ is an arbitrary operator defined on the Hilbert space of the physical system.

It is easy to show from (3.20) or (3.24) that when we define the quantum-filter state $\hat{\sigma}$ of the physical system by

$$(u|\hat{\sigma}|v) = \sum_{j \in \mathcal{S}} p_j \psi_j(-v)\psi_j^*(-u) \tag{4.23}$$

the operation $\hat{\hat{\mathcal{L}}}_k(\Delta_k)$ given by (4.22) satisfies

$$\mathcal{W}_k(\Delta_k) = \mathrm{Tr}[\hat{\hat{\mathcal{L}}}_k(\Delta_k)\hat{\rho}] = \mathrm{Tr}[\hat{\Pi}_k(\Delta_k)\hat{\rho}] \tag{4.24}$$

where $\hat{\Pi}_k(\Delta_k)$ is the operational POVM of the momentum measurement, which is given by (3.20) or (3.24). Therefore we have found that the operation $\hat{\hat{\mathcal{L}}}_k(\Delta_k)$ describes the state change caused by the operational measurement of momentum. The quantum state of the physical system after the operational measurement of momentum is given by the following statistical operator:

$$\hat{\rho}_k(\Delta_k) = \frac{\hat{\hat{\mathcal{L}}}_k(\Delta_k)\hat{\rho}}{\mathrm{Tr}[\hat{\hat{\mathcal{L}}}_k(\Delta_k)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}_k(\Delta_k)\hat{\rho}}{\mathrm{Tr}[\hat{\Pi}_k(\Delta_k)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}_k(\Delta_k)\hat{\rho}}{\mathcal{W}_k(\Delta_k)} \tag{4.25}$$

The operation $\hat{\hat{\mathcal{L}}}_k(\Delta_k)$ and the operational POVM $\hat{\Pi}_k(\Delta_k)$ completely characterize the operational phase-space momentum measurement on the physical system.

4.3. State Change in Simultaneous Measurement of Position and Momentum

We finally consider the state change of the physical system caused by the simultaneous measurement of position and momentum (Arthurs and Kelly, 1965; Busch, 1985; Stenholm, 1992; Braunstein *et al.*, 1991). It will be found later that this measurement is equivalent to the operational phase-space measurement. In this case, we have to prepare two measurement apparatuses as A and B; one "A" for the position measurement and the other "B" for the momentum measurement. Here we denote quantities of the measurement apparatus A with subscript a, such as $\hat{O}_a$ and $|\psi\rangle_a$, and quantities of the measurement apparatus B with subscript b, such as $\hat{O}_b$ and $|\psi\rangle_b$. The interaction between the physical system and the measurement apparatus is assumed to be given by the Hamiltonian,

$$\hat{H}_{\mathrm{int}} = g(\hat{x} \otimes \hat{p}_a \otimes \hat{I}_b + \hat{p} \otimes I_a \otimes \hat{p}_b) \tag{4.26}$$

Suppose that the measurement apparatuses are initially prepared in the quantum state $\hat{\rho}_{ab}$ which is not necessarily factorized into $\hat{\rho}_a \otimes \hat{\rho}_b$. Therefore, after the interaction during $\tau = 1/g$, the composite quantum state $\hat{W}$ of the physical system and the measurement apparatus becomes

$$\hat{W} = \hat{T}(\hat{\rho} \otimes \hat{\rho}_{ab})\hat{T}^{\dagger} \tag{4.27}$$

with

$$\hat{T} = \exp[-i(\hat{x} \otimes \hat{p}_a \otimes \hat{I}_b + \hat{p} \otimes \hat{I}_a \otimes \hat{p}_b)] \tag{4.28}$$

To know the position of the physical system, we read the position value exhibited by the measurement apparatus A, and to know the momentum of the physical system, we read the position value exhibited by the measurement apparatus B. When the position values r and k of the measurement apparatuses A and B belong to the ranges Δ_r and Δ_k, the projection-valued measure $\hat{\mathscr{X}}_{ab}(\Delta_r, \Delta_k)$ of the measurement apparatus becomes

$$\hat{\mathscr{X}}_{ab}(\Delta_r, \Delta_k) = \hat{\mathscr{X}}_a(\Delta_r) \otimes \hat{\mathscr{X}}_b(\Delta_k) = \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk\ |r\rangle_a\ {}_a\langle r| \otimes |k\rangle_b\ {}_b\langle k| \tag{4.29}$$

Thus the nonnormalized statistical operator $\hat{W}(\Delta_r, \Delta_k)$ of the postmeasurement state of the physical system is given by

$$\begin{aligned} \hat{W}(\Delta_r, \Delta_k) &= \mathrm{Tr}_a \mathrm{Tr}_b[(\hat{I} \otimes \hat{\mathscr{X}}_{ab}(\Delta_r, \Delta_k))\hat{W}] \\ &= \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk\ {}_{ab}\langle r, k|\hat{W}|r, k\rangle_{ab} \end{aligned} \tag{4.30}$$

where we set $|r, k\rangle_{ab} = |r\rangle_a \otimes |k\rangle_b$.

To calculate the right-hand side in equation (4.30), we introduce the spectral decomposition of the statistical operator $\hat{\rho}_{ab}$ by

$$\hat{\rho}_{ab} = \sum_{j\in\mathcal{U}} p_j |\psi_j\rangle_{ab}\ {}_{ab}\langle\psi_j| \tag{4.31}$$

where $p_j \geq 0$ and $\Sigma_{j\in u}\, p_j = 1$ are satisfied. Here, $\mathcal{U}$ stands for the spectral set of the statistical operator $\hat{\rho}_{ab}$. If the statistical operator $\hat{\rho}_{ab}$ is factorized into $\hat{\rho}_a \otimes \hat{\rho}_b$, we have

$$j = (j_a, j_b) \in \mathcal{U}_a \times \mathcal{U}_b = \mathcal{U}, \quad p_j = p_{j_a} p_{j_b} \quad \text{and}$$

$$|\psi_j\rangle_{ab} = |\psi_{ja}\rangle_a \otimes |\psi_{jb}\rangle_b$$

Substituting equation (4.31) into equation (4.30), after some calculation, we can obtain the following expression:

$$\begin{aligned} \hat{W}(\Delta_r, \Delta_k) &= \frac{1}{(2\pi)^2} \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk \int_{-\infty}^{\infty} du \\ &\quad \times \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \\ &\quad \times \mathcal{K}_{p\text{-}m}(u, u'; v, v') e^{-i[u(\hat{x}-r)+v(\hat{p}-k)]} \hat{\rho} e^{i[u'(\hat{x}-r)+v'(\hat{p}-k)]} \\ &= \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk \sum_{j\in\mathcal{U}} p_j \hat{\Upsilon}_j(r, k) \hat{\rho} \hat{\Upsilon}_j^{\dagger}(r, k) \end{aligned} \tag{4.32}$$

where we have defined $\mathcal{K}_{\text{p-m}}(u, u'; v, v') = {}_{ab}(u, v|\hat{\rho}_{ab}|u', v')_{ab}$ with $|u, v)_{ab} = |u)_a \otimes |v)_b$, and $\hat{\Upsilon}_j(r, k)$ is an operator defined on the Hilbert space of the physical system,

$$\begin{aligned}\hat{\Upsilon}_j(r, k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; {}_{ab}(u, v|\psi_j\rangle_{ab} \exp[-iu(\hat{x} - r) - iv(\hat{p} - k)] \\ &= \hat{D}(r, k)\hat{\Upsilon}_j\hat{D}^{\dagger}(r, k) \end{aligned} \tag{4.33}$$

Here $\hat{D}(r, k)$ is the displacement operator given by (2.1) and the operator $\hat{\Upsilon}_j$ is given by

$$\begin{aligned}\hat{\Upsilon}_j &= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; {}_{ab}(u, v|\psi_j\rangle_{ab} \exp[-i(u\hat{x} + v\hat{p})] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; {}_{ab}(u, -v|\psi_j\rangle_{ab} D^{\dagger}(v, u) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; {}_{ab}\langle -x, -y|\psi_j\rangle_{ab}\hat{T}(x, y) \end{aligned} \tag{4.34}$$

where the operator $\hat{T}(x, y)$ is the double Fourier transform of the displacement operator $\hat{D}(v, u)$, which is defined by

$$\begin{aligned}\hat{T}(x, y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; \hat{D}(v, u)e^{-i(ux - vy)} \\ &= \frac{1}{\pi}\int_{\xi \in \mathbf{R}^2} d^2\xi \; \hat{D}(\xi)e^{\alpha\xi^* - \alpha^*\xi} = \hat{T}(\alpha) \end{aligned} \tag{4.35}$$

In this equation, the complex variables ξ and α are given by $\xi = (v + iu)/\sqrt{2}$ and $\alpha = (x + iy)/\sqrt{2}$. The operator $\hat{T}(x, y)$ becomes Hermitian and satisfies the following relations (Cahill and Glauber, 1969a, b; Agarwal and Wolf, 1970a–c):

$$\mathrm{Tr}[\hat{T}(x, y)\hat{T}(x', y')] = 2\pi\delta(x - x')\delta(y - y') \tag{4.36}$$

$$\mathrm{Tr}[\hat{T}(\alpha)\hat{T}(\beta)] = \pi\delta^{(2)}(\alpha - \beta) \tag{4.37}$$

Therefore we have found the operation $\hat{\mathcal{L}}(\Delta_r, \Delta_k)$ that describes the state change of the physical system caused by the simultaneous measurement of position and momentum,

$$\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{O} = \frac{1}{(2\pi)^2} \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk \int_{-\infty}^{\infty} du \times \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \times \mathcal{K}_{p\text{-}m}(u, u'; v, v') e^{-i[u(\hat{x}-r)+v(\hat{p}-k)]} \hat{O} e^{i[u'(\hat{x}-r)+v'(\hat{p}-k)]} = \int_{r\in\Delta_r} dr \int_{k\in\Delta_k} dk \sum_{j\in\mathcal{U}} p_j \hat{Y}_j(r, k)\hat{O}\hat{Y}_j^\dagger(r, k) \tag{4.38}$$

where $\hat{O}$ stands for an arbitrary operator defined on the Hilbert space of the physical system.

Next we investigate the relationship between the operation $\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)$ and the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$ of the operational phase-space measurement. For this purpose, we introduce an operator $\hat{\sigma}$ defined on the Hilbert space of the physical system by

$$\hat{\sigma} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \times {}_{ab}\langle -x, -y|\hat{\rho}_{ab}| - x', -y'\rangle_{ab}\, \hat{T}(x', y')\hat{T}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \times {}_{ab}(u, -v|\hat{\rho}_{ab}|u', -v')_{ab}\, \hat{D}(v', u')\hat{D}^\dagger(v, u) \tag{4.39}$$

First it is easy to see from (4.36) that the operator $\hat{\sigma}$ is normalized as

$$\mathrm{Tr}\sigma = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, {}_{ab}\langle x, y|\hat{\rho}_{ab}|x, y\rangle_{ab} = \mathrm{Tr}_a\mathrm{Tr}_b\hat{\rho}_{ab} = 1 \tag{4.40}$$

Next we show that $\hat{\sigma}$ is a nonnegative Hermitian operator. To this end, we calculate $\langle\phi|\hat{\sigma}|\phi\rangle$ for an arbitrary state vector $|\phi\rangle$ in the Hilbert space of the physical system,

$$\langle\phi|\hat{\sigma}|\phi\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \times {}_{ab}(u, -v|\hat{\rho}_{ab}|u', -v')_{ab}\, \langle\phi|\hat{D}(v', u')\hat{D}^\dagger(v, u)|\phi\rangle = \frac{1}{2\pi} \sum_{j\in\mathcal{U}} p_j \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \times {}_{ab}(u, -v|\psi_j\rangle_{ab}\, {}_{ab}\langle\psi_j|u', -v')_{ab}\, \langle\phi|\hat{D}(v', u')\hat{D}^\dagger(v, u)|\phi\rangle \tag{4.41}$$

where we have used the spectral decomposition of the statistical operator $\hat{\rho}_{ab}$ of the measurement apparatus given by (4.31). We now define a state vector $|\Psi_j\rangle$ in the Hilbert space of the physical system by

$$|\Psi_j\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \hat{D}^{\dagger}(v, u)|\phi\rangle\, {}_{ab}(u, -v|\psi_j\rangle_{ab} \tag{4.42}$$

Using this state vector, we finally obtain the inequality

$$\langle\phi|\hat{\sigma}|\phi\rangle = \sum_{j\in\mathcal{U}} p_j\langle\Psi_j|\Psi_j\rangle \geq 0 \tag{4.43}$$

which indicates that the operator $\hat{\sigma}$ is nonnegative. Furthermore, it is clear from the definition that $\hat{\sigma}$ is a Hermitian operator. Thus we have found that the operator $\hat{\sigma}$ given by (4.39) becomes the statistical operator defined on the Hilbert space of the physical system and we can consider the quantum state described the statistical operator $\hat{\sigma}$ the quantum-filter state of the physical system.

Therefore, using equations (4.33), (4.34), (4.38), and (4,39), we can obtain the relation

$$\begin{aligned}\mathcal{W}(\Delta_r, \Delta_k) &= \mathrm{Tr}[\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{\rho}] = \mathrm{Tr}[\hat{\Pi}(\Delta_r, \Delta_k)\hat{\rho}] \\ &= \frac{1}{2\pi}\int_{r\in\Delta_r} dr \int_{k\in\Delta_k} \mathrm{Tr}[\hat{\rho}\hat{D}(r, k)\hat{\sigma}\hat{D}^{\dagger}(r, k)]\end{aligned} \tag{4.44}$$

where $\hat{\Pi}(\Delta_r, \Delta_k)$ is the operational POVM given by (3.8). This result indicates that the operation $\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)$ given by (4.38) describes the state change of the physical system caused by the operational phase-space measurement of position and momentum. The quantum state of the physical system after the operational phase-space measurement is given by the following statistical operator:

$$\hat{\rho}(\Delta_r, \Delta_k) = \frac{\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{\rho}}{\mathrm{Tr}[\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{\rho}}{\mathrm{Tr}[\hat{\Pi}(\Delta_r, \Delta_k)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{\rho}}{\mathcal{W}(\Delta_r, \Delta_k)} \tag{4.45}$$

The operational phase-space measurement is completely characterized by the operation $\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)$ and the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$.

Let us consider the case that the quantum states of the measurement apparatus are prepared in the squeezed-vacuum states,

$$\hat{\rho}_{ab} = |\gamma\rangle_{aa}\langle\gamma| \otimes |\gamma'\rangle_{bb}\langle\gamma'| \tag{4.46}$$

where the squeezing parameters γ and γ' are assumed to be real. In this case, we obtain

$${}_a\langle -x|\gamma\rangle_a\, {}_b\langle -y|\gamma'\rangle_b = \frac{1}{\sqrt{\pi e^{\gamma+\gamma'}}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{e^{2\gamma}} + \frac{y^2}{e^{2\gamma'}}\right)\right] \tag{4.47}$$

In particular, when we set the squeezing parameters $\gamma = \gamma' = \frac{1}{2}\ln 2$, the operator $\hat{Y}_j$ given by (4.34) becomes

$$\hat{Y} = \frac{1}{\pi\sqrt{2\pi}} \int_{\alpha \in \mathbb{R}^2} d^2\alpha\ \hat{T}(\alpha) e^{-|\alpha|^2/2}$$
$$= \frac{1}{\pi\sqrt{2\pi}} \int_{\alpha \in \mathbb{R}^2} d^2\alpha\ \hat{D}(\alpha) e^{-|\alpha|^2/2}$$
$$= \frac{1}{\sqrt{2\pi}} |0\rangle\langle 0| \tag{4.48}$$

It should be noted here that the index j of the summation does not appear, since the quantum state $\hat{\rho}_{ab}$ of the measurement apparatus is pure. Thus the operation $\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)$, the operational POVM $\hat{\Pi}(\Delta_r, \Delta_k)$, and the operational phase-space probability $\mathcal{W}(\Delta_r, \Delta_k)$ of the simultaneous measurement of position and momentum are given by

$$\hat{\hat{\mathcal{L}}}(\Delta_r, \Delta_k)\hat{O} = \frac{1}{\pi} \int_{\mu \in \tilde{\Delta}_r \times \tilde{\Delta}_k} d^2\mu\ |\mu\rangle\langle\mu|\hat{O}|\mu\rangle\langle\mu| \tag{4.49}$$

$$\hat{\Pi}(\Delta_r, \Delta_k) = \frac{1}{\pi} \int_{\mu \in \tilde{\Delta}_r \times \tilde{\Delta}_k} d^2\mu\ |\mu\rangle\langle\mu| \tag{4.50}$$

$$\mathcal{W}(\Delta_r, \Delta_k) = \frac{1}{\pi} \int_{\mu \in \tilde{\Delta}_r \times \tilde{\Delta}_k} d^2\mu\ \langle\mu|\hat{\rho}|\mu\rangle \tag{4.51}$$

where $|\mu\rangle$ is the coherent state with $\mu = (r + ik)/\sqrt{2}$. The operational POVM given by equation (4.50) is equivalent to that of the ideal optical heterodyne detection (Yuen and Shapiro, 1978, 1980; Shapiro *et al.*, 1979; Busch *et el.*, 1995).

Before closing this section, we consider the case that we do not read the result of the position (momentum) measurement in the simultaneous measurement of position and momentum. When we do not read the outcome exhibited by the measurement apparatus B, the operation $\mathcal{L}_r(\Delta_r)$ of the marginal position measurement becomes

$$\hat{\hat{\mathcal{L}}}(\Delta_r)\hat{O} = \hat{\hat{\mathcal{L}}}(\Delta_r, \mathbb{R})\hat{O}$$
$$= \frac{1}{2\pi} \int_{r \in \Delta_r} dr \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du'\ \mathcal{K}_{p\text{-}m}(u, u'; -v, -v)$$
$$\times e^{-ir\hat{p}} \hat{D}^\dagger(v, u) e^{ir\hat{p}} \hat{O} e^{-ir\hat{p}} \hat{D}(v, u') e^{ir\hat{p}} \tag{4.52}$$

and the corresponding marginal POVM $\hat{\Pi}_r(\Delta_r)$ is given by

$$\hat{\Pi}_r(\Delta_r) = \int_{r\in\Delta_r} dr \int_{-\infty}^{\infty} dx\, f_{p\text{-}m}(x - r)|x\rangle\langle x| \tag{4.53}$$

where $f_{\text{p-m}}(x) = {}_a\langle -x|\hat{\sigma}_a| - x\rangle_a$ is the filter function, and the effective statistical operator $\hat{\sigma}_a$ of the measurement apparatus A is given by

$$\hat{\sigma}_a = \text{Tr}_b \left[\exp\left(-\frac{1}{2} i\hat{p}_a \otimes \hat{p}_b\right)\hat{\rho}_{ab} \exp\left(\frac{1}{2} i\hat{p}_a \otimes \hat{p}_b\right)\right] \tag{4.54}$$

The unitary operator $\exp(-\frac{1}{2} i\hat{p}_a \otimes \hat{p}_b)$ represents the quantum correlation between the measurement apparatuses A and B through the measured physical system. Mathematically, this unitary operator appears when we decompose the unitary operator $\hat{T}$ given by (4.28) into

$$\hat{T} = \exp(-i\hat{p} \otimes \hat{I}_a \otimes \hat{p}_b) \exp(-i\hat{x} \otimes \hat{p}_a \otimes \hat{I}_b) \exp\left(-\frac{1}{2} i\hat{I} \otimes \hat{p}_a \otimes \hat{p}_b\right)$$

to trace out the variables of the measurement apparatus B. When we define the quantum-filter state $\hat{\sigma}$ by the relation $\langle x|\hat{\sigma}|y\rangle = {}_a\langle -y|\hat{\sigma}_a| - x\rangle_a$, equation (4.53) becomes equivalent to equation (3.19).

On the other hand, when we do not read the outcome exhibited by the measurement apparatus A, the operation $\hat{\hat{\mathcal{L}}}_k(\Delta_k)$ of the marginal momentum measurement becomes

$$\begin{aligned}\hat{\hat{\mathcal{L}}}_k(\Delta_r)\hat{O} &= \hat{\hat{\mathcal{L}}}(\mathbb{R}, \Delta_k)\hat{O} \\ &= \frac{1}{2\pi} \int_{k\in\Delta_k} dk \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dv'\, \mathcal{K}_{\text{p-m}}(u, u; -v, -v') \\ &\quad \times e^{ik\hat{x}}\hat{D}^{\dagger}(v, u)e^{-ik\hat{x}}\hat{O}e^{ik\hat{x}}\hat{D}(v', u)e^{-ik\hat{x}} \end{aligned} \tag{4.55}$$

and the corresponding marginal POVM $\hat{\Pi}_k(\Delta_k)$ is given by

$$\hat{\Pi}_k(\Delta_k) = \int_{k\in\Delta_k} dk \int_{-\infty}^{\infty} dp\, g_{\text{p-m}}(p - k)|p)(p| \tag{4.56}$$

where $g_{\text{p-m}}(p) = {}_a(-p|\hat{\sigma}_b| - p)_a$ is the filter function and the effective statistical operator $\hat{\sigma}_b$ of the measurement apparatus B is given by

$$\hat{\sigma}_b = \text{Tr}_a \left[\exp\left(\frac{1}{2} i\hat{p}_a \otimes \hat{p}_b\right)\hat{\rho}_{ab} \exp\left(-\frac{1}{2} i\hat{p}_a \otimes \hat{p}_b\right)\right] \tag{4.57}$$

The unitary operator $\exp(\frac{1}{2} i\hat{p}_a \otimes \hat{p}_b)$ also represents the quantum correlation between the measurement apparatuses A and B through the measured physical system, which appears when the unitary operator $\hat{T}$ is decomposed into

$$\hat{T} = \exp(-i\hat{x} \otimes \hat{p}_a \otimes \hat{I}_b) \exp(-i\hat{p} \otimes \hat{I}_a \otimes \hat{p}_b) \exp\left(\frac{1}{2} i\hat{I} \otimes \hat{p}_a \otimes \hat{p}_b\right)$$

to trace out the variables of the measurement apparatus A. Therefore, introducing the quantum-filter state $\hat{\sigma}$ by the relation $(u|\hat{\sigma}|v) = {}_b(-v|\hat{\sigma}_b|-u)_b$, we find that (4.56) is equivalent to (3.20).

In this section, we have assumed the interaction Hamiltonians (4.3), (4.17), and (4.26) between the physical system and the measurement apparatus to obtain the operations that describe the state change of the physical system caused by the operational phase-space measurements. If we assume a different interaction Hamiltonian, we obtain a different operation that corresponds to the operational POVM.

5. ENTROPY AND INFORMATION IN OPERATIONAL PHASE-SPACE MEASUREMENT

5.1. Position and Momentum Measurements

When we perform a quantum measurement on a physical system to obtain some information about its quantum state, the measured quantum state of the physical system is changed into another quantum state by the effect of the quantum measurement. Thus the entropy of the quantum state also changes in the quantum measurement. In Section 4, assuming the interaction Hamiltonian between the physical system and the measurement apparatus, we obtained the operation that describes the state change caused by the operational phase-space measurement. In this section, using the results, we will investigate the entropy change of the quantum state and the information gain in the operational phase-space measurement. For this purpose, we apply the measurement entropy (Ballan *et el.*, 1986), but not the von Neumann entropy, since we would like to know the information about the observable quantity, such as the position and momentum of the physical system, in the operational phase-space measurement. Furthermore, we consider the relationship between the entropy change and the Shannon mutual information (the mean information content) extracted from the outcomes exhibited by the measurement apparatus (Shannon, 1948a, b; Brillouin, 1956; Majernik, 1970, 1973; Cover and Thomas, 1991). The Shannon mutual information serves as a measure of the statistical linkage between the premeasurement system and the measurement apparatus.

For the physical system prepared in the quantum state $\hat{\rho}$, when we observe a quantity described by the Hermitian operator $\hat{O}$, we obtain the value a as the measurement outcome with probability or probability density

$$P(a) = \langle\psi_a|\hat{\rho}|\psi_a\rangle \tag{5.1}$$

where $|\psi_a\rangle$ is the eigenstate of the operator $\hat{O}$ with eigenvalue a, that is, $\hat{O}|\psi_a\rangle = a|\psi_a\rangle$. If the operator $\hat{O}$ has a discrete spectrum, the measurement entropy is given by

$$S = -\sum_{a\in\mathcal{A}} P(a) \ln P(a) \tag{5.2}$$

where $\mathcal{A}$ stands for the spectral set of the operator $\hat{O}$ and we measure the entropy in *nats*. If the operator $\hat{O}$ has a continuous spectrum, the measurement entropy becomes

$$S = -\int_{a\in\mathcal{A}} da\, P(a) \ln P(a) \tag{5.3}$$

which is called the differential entropy (Cover and Thomas, 1991). It is important to note that the entropy given by (5.2) is nonnegative, while the differential entropy defined by (5.3) can take negative values in some cases. In this section, we use the differential entropy, since we consider measurements of position and momentum observables which have continuous spectra.

Suppose that the quantum state $\hat{\rho}$ of the physical system is changed into another quantum state $\hat{\rho}'$ by the effect of the quantum measurement. Then the entropy of the system also changes as $S \to S'$. The entropy change $\Delta S = S - S'$ of the quantum state is considered the information I gained by the quantum measurement (Brillouin, 1956)

$$I = \Delta S = S - S' \tag{5.4}$$

In this section, we define the entropy change ΔS by subtracting the postmeasurement entropy from the premeasurement entropy.

We first consider the entropy change or the information gain in the operational phase-space measurement of position. Before the measurement, the physical system is prepared in the quantum state $\hat{\rho}$. Thus the premeasurement entropy defined on the set of position probability distributions is given by

$$S_r^{\text{in}} = -\int_{-\infty}^{\infty} dx\, P_{\text{in}}(x) \ln P_{\text{in}}(x) \tag{5.5}$$

with

$$P_{\text{in}}(x) = \langle x|\hat{\rho}|x\rangle \tag{5.6}$$

After the position measurement, when we obtain the measurement outcome r which belongs to the range Δ_r, the quantum state of the physical system is given by (4.16). In this case, the conditional probability density of position is calculated as

$$P_r^{\text{out}}(x|r) = \langle x|\hat{\rho}_r(r)|x\rangle \tag{5.7}$$

which is conditioned by the measurement outcome r. Here the statistical operator $\hat{\rho}_r(r)$ is given by $\hat{\rho}_r(r) = \lim_{|\Delta_r|\to 0} \hat{\rho}_r(\Delta_r)$, where $|\Delta_r|$ is the width of the interval Δ_r. Then we obtain from (4.12) and (4.13),

$$\hat{\rho}_r(r) = \frac{\hat{\hat{\mathcal{L}}}_r(r)\hat{\rho}}{\mathrm{Tr}[\hat{\hat{\mathcal{L}}}_r(r)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}_r(r)\hat{\rho}}{\mathrm{Tr}[\hat{\Pi}_r(r)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}_r(r)\hat{\rho}}{\mathcal{W}_r(r)} \tag{5.8}$$

where the operation $\hat{\hat{\mathcal{L}}}_r(r)$, the operational POVM $\hat{\Pi}_r(r)$, and the operational phase-space probability density $\mathcal{W}_r(r)$ are respectively given by

$$\hat{\hat{\mathcal{L}}}_r(r)\hat{O} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathcal{K}_p(r - x, r - y)|x\rangle\langle x|\hat{O}|y\rangle\langle y| \tag{5.9}$$

$$\hat{\Pi}_r(r) = \int_{-\infty}^{\infty} dx\, \mathcal{K}_p(r - x, r - x)|x\rangle\langle x| \tag{5.10}$$

$$\mathcal{W}_r(r) = \int_{-\infty}^{\infty} dx\, \mathcal{K}_p(r - x, r - x)\langle x|\hat{\rho}|x\rangle \tag{5.11}$$

with $\mathcal{K}_p(x, y) = {}_a\langle x|\hat{\rho}_a|y\rangle_a$.

Thus the conditional entropy for the given measurement outcome r is obtained by

$$S_r^{\text{out}}(r) = -\int_{-\infty}^{\infty} dx\, P_r^{\text{out}}(x|r) \ln P_r^{\text{out}}(x|r) \tag{5.12}$$

Since the probability density that we obtain the measurement outcome r is given by (5.11), the average value of the conditional entropy of the physical system after the operational position measurement is calculated as

$$S_r^{\text{out}} = \int_{-\infty}^{\infty} dr\, \mathcal{W}_r(r) S_r^{\text{out}}(r) \tag{5.13}$$

Therefore, from equations (5.4), (5.5) and (5.13), the information I_r obtained by the operational position measurement is given by the entropy change,

$$I_r = \Delta S_r = S_r^{\text{in}} - S_r^{\text{out}} \tag{5.14}$$

We now obtain the average value S_r^{out} of the conditional entropy of the physical system after the measurement. Since we obtain from (5.9)

$$\langle x|\hat{\hat{\mathcal{L}}}_r(r)\hat{\rho}|x\rangle = {}_a\langle r - x|\hat{\rho}_a|r - x\rangle_a\langle x|\hat{\rho}|x\rangle \tag{5.15}$$

we can calculate the average value S_r^{out} of the conditional entropy from (5.7), (5.8), and (5.11)–(5.13),

$$\begin{aligned} S_r^{\text{out}} &= -\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dx\langle x|\hat{\hat{\mathcal{L}}}_r(r)\hat{\rho}|x\rangle \ln\left[\frac{\langle x|\hat{\hat{\mathcal{L}}}_r(r)\hat{\rho}|x\rangle}{\mathcal{W}_r(r)}\right] \\ &= -\int_{-\infty}^{\infty} dx\langle x|\hat{\rho}|x\rangle \ln\langle x|\hat{\rho}|x\rangle - \int_{-\infty}^{\infty} dx\ {}_a\langle x|\hat{\rho}_a|x\rangle_a \ln {}_a\langle x|\hat{\rho}_a|x\rangle_a \\ &\quad + \int_{-\infty}^{\infty} dr\, \mathcal{W}_r(r) \ln \mathcal{W}_r(r) \\ &= S_r^{\text{in}} + S_r^{(a)} - H_r \end{aligned} \tag{5.16}$$

where S_r^{in} is the initial entropy of the physical system and $S_r^{(a)}$ is the initial entropy of the measurement apparatus prepared in the quantum state $\hat{\rho}_a$,

$$S_r^{(a)} = -\int_{-\infty}^{\infty} dx\ {}_a\langle x|\hat{\rho}_a|x\rangle_a \ln {}_a\langle x|\hat{\rho}_a|x\rangle_a \tag{5.17}$$

Furthermore, the quantity H_r in (5.16) represents the differential entropy calculated from the operational probability density $\mathcal{W}_r(r)$ of the measurement outcome,

$$H_r = -\int_{-\infty}^{\infty} dr\, \mathcal{W}_r(r) \ln \mathcal{W}_r(r) \tag{5.18}$$

Therefore we obtain the expression for the information gain I_r in the operational phase-space measurement of position,

$$I_r = H_r - S_r^{(a)} \tag{5.19}$$

This result indicates that the information about the physical system extracted from the operational position measurement is equal to the difference between the entropy calculated from the measurement outcomes exhibited by the measurement apparatus and the initial entropy of the measurement apparatus. It is easy to see that we have to use the measurement apparatus that has the smaller entropy to obtain more information about the measured physical system.

In the same way, we can obtain the information I_k about the physical system gained in the operational momentum measurement. The information I_k is calculated by

$$I_k = \Delta S_k = S_k^{\text{in}} - S_k^{\text{out}} \tag{5.20}$$

where S_k^{in} is the entropy in the initial quantum state $\hat{\rho}$ of the physical system and S_k^{out} is the average value of the conditional entropy of the physical system after the measurement, which are given by

$$S_k^{\text{in}} = -\int_{-\infty}^{\infty} dp (p|\hat{\rho}|p) \ln(p|\hat{\rho}|p) \tag{5.21}$$

$$S_k^{\text{out}} = \int_{-\infty}^{\infty} dk\, \mathcal{W}_k(k) S_k^{\text{out}}(k) \tag{5.22}$$

with

$$S_k^{\text{out}}(k) = -\int_{-\infty}^{\infty} dp\, P_k^{\text{out}}(p|k) \ln P_k^{\text{out}}(p|k) \tag{5.23}$$

Here, $\mathcal{W}_k(k)$ is the operational probability density that we obtain the measurement outcome k in the momentum measurement,

$$\mathcal{W}_k(k) = \int_{-\infty}^{\infty} dp\, \mathcal{K}_m(k - p, k - p)(p|\hat{\rho}|p) \tag{5.24}$$

where $\mathcal{K}_m(u, v) = {}_a(u|\hat{\rho}_a|v)_a$, and $P_k^{\text{out}}(p|k)$ is the conditional probability density for the given measurement outcome k,

$$P_k^{\text{out}}(p|k) = (p|\hat{\rho}_k(k)|p) \tag{5.25}$$

where the statistical operator $\hat{\rho}_k(k)$ of the postmeasurement state of the physical system when we obtain the measurement outcome k is given by

$$\hat{\rho}_k(k) = \frac{\hat{\hat{\mathcal{L}}}_k(k)\hat{\rho}}{\text{Tr}[\hat{\hat{\mathcal{L}}}_k(k)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}_k(k)\hat{\rho}}{\text{Tr}[\hat{\Pi}_k(k)\hat{\rho}]} = \frac{\hat{\hat{\mathcal{L}}}_k(k)\hat{\rho}}{\mathcal{W}_k(k)} \tag{5.26}$$

with

$$\hat{\hat{\mathcal{L}}}_k(k)\hat{O} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \mathcal{K}_m(k - u, k - v)|u)(u|\hat{O}|v)(v| \tag{5.27}$$

$$\hat{\Pi}_k(k) = \int_{-\infty}^{\infty} du\, \mathcal{K}_m(k - u, k - u)|u)(u| \tag{5.28}$$

These equations are derived from (4.22), (4.24), and (4.25).

Therefore we can obtain the expression for the information I_k extracted from the outcomes of the operational momentum measurement,

$$I_k = H_k - S_k^{(a)} \tag{5.29}$$

where H_k is the differential entropy calculated from the operational probability density $\mathcal{W}_k(k)$ of the measurement outcome and $S_k^{(a)}$ is the initial entropy of the measurement apparatus in the quantum state $\hat{\rho}_a$,

$$H_k = -\int_{-\infty}^{\infty} dk\, \mathcal{W}_k(k) \ln \mathcal{W}_k(k) \tag{5.30}$$

$$S_k^{(a)} = -\int_{-\infty}^{\infty} dp\, {}_a\langle p|\hat{\rho}_a|p\rangle_a \ln {}_a\langle p|\hat{\rho}_a|p\rangle_a \tag{5.31}$$

The result indicates that the information about the physical system obtained by the operational phase-space measurement of momentum is equal to the difference between the entropy calculated from the measurement outcomes exhibited by the measurement apparatus and the initial entropy of the measurement apparatus.

5.2. Properties of the Information in the Quantum Measurement

We now consider the properties of the information, I_r and I_k, given by equation (5.19) and (5.29). First we consider the case that the accuracy δx of the position measurement is very high such that the condition $\delta x << c|d\langle r|\hat{\rho}|r\rangle/dr|$ can be satisfied, where c is some constant factor. In this case, we may assume that the quantum state of the measurement apparatus is approximated by ${}_a\langle x|\hat{\rho}_a|x\rangle_a = (\delta x)^{-1}$ for $x \in [-\delta x/2, \delta x/2)$ and ${}_a\langle x|\hat{\rho}_a|x\rangle_a = 0$ for $x \notin [-\delta x/2, \delta x/2)$. Thus we can approximate the operational probability density $\mathcal{W}_r(r)$ given by (5.11),

$$\mathcal{W}_r(r) \approx \int_{\delta x/2}^{\delta x/2} dx \frac{1}{\delta x} \langle r - x|\hat{\rho}|r - x\rangle \approx \langle r|\hat{\rho}|r\rangle \approx \frac{1}{\delta x} P(j) \tag{5.32}$$

where $P(j)$ is the probability that the measurement outcome r belongs to the interval of the real axis, $[r_j - \delta x/2, r_j + \delta x/2)$, and the discrete variable r_j satisfies the relation $r_{j+1} - r_j = \delta x$ for all j's. Note that the quantity $(\delta x)^{-1}$ appeared in the last term of (5.32) is due to the normalization condition, $\Sigma_j P(j) = \int_{-\infty}^{\infty} dr\, \mathcal{W}_r(r) = 1$. Thus the differential entropy H_r given by (5.18) becomes

$$\begin{aligned} H_r &\approx -\int_{-\infty}^{\infty} dr\, \langle r|\hat{\rho}|r\rangle \ln\langle r|\hat{\rho}|r\rangle \\ &\approx -\sum_i \int_{r_j-\delta x/2}^{r_j+\delta x/2} dr \frac{P(r_j)}{\delta x} \ln\left[\frac{P(r_j)}{\delta x}\right] \\ &= -\sum_j P(j) \ln P(j) + \ln(\delta x) \end{aligned} \tag{5.33}$$

and the initial entropy $S_r^{(a)}$ of the measurement apparatus is approximated by

$$S_r^{(a)} \approx -\int_{-\delta x/2}^{\delta x/2} dx \, \frac{1}{\delta x} \ln\left(\frac{1}{\delta x}\right) = \ln(\delta x) \tag{5.34}$$

Therefore the information I_r about the physical system gained in the operational position measurement is obtained from equations (5.33) and (5.34),

$$I_r = H_r - S_r^{(a)} \approx -\sum_j P(j) \ln P(j) \tag{5.35}$$

This result indicates that when the accuracy of the position measurement is very high, the information I_r is equal to the Shannon entropy (the information-theoretic entropy) calculated from the probability of the discretized position of the physical system in the initial quantum state $\hat{\rho}$. The same result can be obtained for the operational momentum measurement.

In quantum mechanics the position and momentum of a physical system cannot be measured simultaneously with very high accuracy due to the Heisenberg uncertainty relation for position and momentum, which is derived from the noncommutativity of position and momentum operators, namely, $[\hat{x}, \hat{p}] = i$ ($\hbar = 1$). The uncertainty relation can be interpreted by means of the position and momentum entropies, which is called the entropic uncertainty relation (Bialynicki-Birula and Mycielski, 1975; Maassen and Uffink, 1988; Partovi, 1983). In our case, the entropic uncertainty relation of the measurement apparatus is given by

$$S_r^{(a)} + S_k^{(a)} \geq \ln(\pi e) \tag{5.36}$$

which is equivalent to the Heisenberg uncertainty relation, $\Delta x_a \Delta p_a \geq 1/2$. For example, when the measurement apparatus is prepared in the squeezed state with real squeezing parameter γ, the probability distributions of position and momentum become

$$_a\langle x|\hat{\rho}_a|x\rangle_a = \frac{1}{e^{\gamma}\sqrt{\pi}} \exp\left[-\left(\frac{x-\bar{x}}{e^{\gamma}}\right)^2\right] \tag{5.37}$$

$$_a(p|\hat{\rho}_a|p)_a = \frac{1}{e^{-\gamma}\sqrt{\pi}} \exp\left[-\left(\frac{p-\bar{p}}{e^{-\gamma}}\right)^2\right] \tag{5.38}$$

where $\bar{x}$ and $\bar{p}$ are the real and imaginary parts of the complex amplitude of the squeezed state. Then we obtain the differential entropies of position and momentum,

$$S_r^{(a)} = \tfrac{1}{2}\ln(\pi e^{1+2\gamma}), \qquad S_k^{(a)} = \tfrac{1}{2}\ln(\pi e^{1-2\gamma}) \tag{5.39}$$

which yields the equality $S_r^{(a)} + S_k^{(a)} = \ln(\pi e)$, since the squeezed state with real squeezing parameter is the minimum-uncertainty state in which $\Delta x_a \Delta p_a = 1/2$ is attained.

On the other hand, it is shown that the differential entropies of position and momentum $S_r^{(a)}$ and $S_k^{(a)}$ satisfy the inequalities (Hall, 1995),

$$S_r(a) \leq \tfrac{1}{2} \ln(2\pi e) + \ln \Delta x_a \tag{5.40}$$

$$S_k(a) \leq \tfrac{1}{2} \ln(2\pi e) + \ln \Delta p_a \tag{5.41}$$

which can be derived by the variational method. Therefore, we obtain the inequality

$$S_r^{(a)} + S_k^{(a)} \leq \ln(2\pi e) + \ln(\Delta x_a \Delta p_a) \tag{5.42}$$

Using the relations (5.36) and (5.42), we find that the information I_r and I_k obtained by the operational position and momentum measurements satisfies

$$H_r + H_k - \ln(2\pi e \Delta x_a \Delta p_a) \leq I_r + I_k \leq H_r + H_k - \ln(\pi e) \tag{5.43}$$

It should be noted here that the uncertainty product $\Delta x_a \Delta p_a$ is calculated in the initial quantum state $\hat{\rho}_a$ of the measurement apparatus. In particular, when we use the measurement apparatus prepared in the minimum-uncertainty state which yields the equality $\Delta x_a \Delta p_a = 1/2$, we obtain the relation

$$I_r + I_k = H_r + H_k - \ln(e\pi) \tag{5.44}$$

Let us now consider the case that the measurement apparatus is prepared in a thermal state where the average value of the thermal photon number is given by $\bar{n}$. In this case, we obtain the position and momentum probability distributions of the measurement apparatus,

$$_a\langle x|\hat{\rho}_a|x\rangle_a = \frac{1}{\sqrt{\pi(1+2\bar{n})}} \exp\left(-\frac{x^2}{1+2\bar{n}}\right) \tag{5.45}$$

$$_a\langle p|\hat{\rho}_a|p\rangle_a = \frac{1}{\sqrt{\pi(1+2\bar{n})}} \exp\left(-\frac{p^2}{1+2\bar{n}}\right) \tag{5.46}$$

Thus the initial entropies $S_r^{(a)}$ and $S_k^{(a)}$ of the measurement apparatus are calculated as

$$S_r^{(a)} = S_k^{(a)} = \tfrac{1}{2} \ln[\pi e(1+2\bar{n})] \tag{5.47}$$

To proceed further, suppose that before the measurement, the physical system is in the coherent state $|\alpha\rangle$ with complex amplitude $\alpha = (q + ip)/\sqrt{2}$. Then, substituting (5.45) and (5.46) into (5.11) and (5.24), we obtain the operational probability densities $\mathcal{W}_r(r)$ and $\mathcal{W}_k(k)$ of the measurement outcomes

$$\mathcal{W}_r(r) = \frac{1}{\sqrt{2\pi(1+\bar{n})}} \exp\left[-\frac{(r-q)^2}{2(1+\bar{n})}\right] \tag{5.48}$$

$$\mathcal{W}_k(k) = \frac{1}{\sqrt{2\pi(1+\bar{n})}} \exp\left[-\frac{(k-p)^2}{2(1+\bar{n})}\right] \tag{5.49}$$

which yields the entropies for the results of the operational phase-space measurements,

$$H_r = H_k = \tfrac{1}{2}\ln[2\pi e(1+\bar{n})] \tag{5.50}$$

Therefore we find the information about the physical system obtained by the position and momentum measurements from (5.47) and (5.50),

$$I_r = I_k = \frac{1}{2}\ln\left[\frac{2(1+\bar{n})}{1+2\bar{n}}\right] \tag{5.51}$$

which satisfies inequality $0 \leq I_r\ (I_k) \leq \ln\sqrt{2}$. The maximum value of the information that we can obtain by the operational measurement is 0.5 bits, which is attained by the measurement apparatus prepared in the vacuum state ($\bar{n} = 0$). Note that the fluctuations of position and momentum in the thermal state of the measurement apparatus become $\Delta x_a = \Delta p_a = \sqrt{\bar{n} + 1/2}$. Thus we obtain the relation,

$$H_r + H_k - \ln[2\pi e \Delta x_a \Delta p_a] = \ln\left[\frac{2(1+\bar{n})}{1+2\bar{n}}\right] = I_r + I_k \tag{5.52}$$

which means that the lower bound in (5.43) is attained in this measurement.

If we know only the average value $\bar{m}$ of the photon number of the physical system before the measurement, the quantum state $\hat{\rho}$ of the physical system is estimated by, according to the maximum-entropy principle (Jaynes, 1957a, b),

$$\hat{\rho} = \frac{1}{1+\bar{m}} \sum_{n=0}^{\infty} \left(\frac{\bar{m}}{1+\bar{m}}\right)^n |n\rangle\langle n| \tag{5.53}$$

In this case, the operational probability densities $\mathcal{W}_r(r)$ and $\mathcal{W}_k(k)$ of the measurement outcomes are calculated as

$$\mathcal{W}_r(r) = \frac{1}{\sqrt{2\pi(1+\bar{m}+\bar{n})}} \exp\left[-\frac{r^2}{2(1+\bar{m}+\bar{n})}\right] \tag{5.54}$$

$$\mathcal{W}_k(k) = \frac{1}{\sqrt{2\pi(1+\bar{m}+\bar{n})}} \exp\left[-\frac{k^2}{2(1+\bar{m}+\bar{n})}\right] \tag{5.55}$$

Therefore we obtain the information I_r and I_k about the physical system extracted from the outcomes of the operational position and momentum measurements,

$$I_r = I_k = \frac{1}{2} \ln\left[\frac{2(1 + \overline{m} + \overline{n})}{1 + 2\overline{n}}\right] \tag{5.56}$$

If we can ignore the effect of the quantum fluctuations of the physical system and the measurement apparatus ($\overline{m} >> 1, \overline{n} >> 1$), equation (5.56) becomes

$$I_r = I_k \approx \frac{1}{2} \ln\left(1 + \frac{\overline{m}}{\overline{n}}\right) \tag{5.57}$$

This result is similar to the classical capacity $C \propto \ln(1 + \mathcal{S}/\mathcal{N})$ for the Gaussian communication channel, where the channel capacity is the maximum value of the mutual information (Shannon, 1948b; Cover and Thomas, 1991; Caves and Drummond, 1994) and $\mathcal{S}/\mathcal{N}$ $(= \overline{m}/\overline{n})$ represents the signal-to-noise ratio in the communication channel. The position or momentum measurement corresponds to the homodyne detection in quantum optical systems.

Finally we investigate the relation of the information I_r (I_k) calculated from the entropy change of the physical system to the Shannon mutual information $\mathcal{I}_r$ ($\mathcal{I}_k$) obtained from the measurement outcomes. The Shannon mutual information $\mathcal{I}_r$ for the position measurement is given by (Shannon, 1948a, b; Brillouin, 1956; Cover and Thomas, 1991)

$$\mathcal{I}_r = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; P(x|y)P_{\text{in}}(y) \ln\left[\frac{P(x|y)}{P_{\text{out}}(x)}\right] \tag{5.58}$$

where $P_{\text{in}}(x)$ and $P_{\text{out}}(x)$ are the input and output probability densities, and $P(x|y)$ is the conditional probability density which connects the input probability density with the output probability density,

$$P_{\text{out}}(x) = \int_{-\infty}^{\infty} dy \; P(x|y)P_{\text{in}}(y) \tag{5.59}$$

The Shannon mutual information $\mathcal{I}_k$ obtained in the momentum measurement is defined in the same way.

In our case of the operational phase-space measurement, the input probability corresponds to the initial (or the premeasurement) position probability of the physical system in the quantum state $\hat{\rho}$ and the output probability to the operational position probability. Thus we find that $P_{\text{in}}(x) = \langle x|\hat{\rho}|x\rangle$ and $P_{\text{out}}(x) = \mathcal{W}_r(x)$. Furthermore, comparing (5.11) with (5.59), it is easy to see that the conditional probability density is given by $P(x|y) = {}_a\langle x - y|\hat{\rho}_a|x -$

$y\rangle_a$. Therefore, when we express the average value S_r^{out} of the conditional entropy of the physical system in terms of $P_{\text{in}}(x)$, $P_{\text{out}}(x)$, and $P(x|y)$, we obtain

$$\begin{aligned} S_r^{\text{out}} &= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, P(x|y)P_{\text{in}}(y) \ln\left[\frac{P(x|y)P_{\text{in}}(y)}{P_{\text{out}}(x)}\right] \\ &= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, P(x|y)P_{\text{in}}(y) \ln\left[\frac{P(x|y)}{P_{\text{out}}(x)}\right] \\ &\quad - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, P(x|y)P_{\text{in}}(y) \ln P_{\text{in}}(y) \\ &= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, P(x|y)P_{\text{in}}(y) \ln\left[\frac{P(x|y)}{P_{\text{out}}(x)}\right] - \int_{-\infty}^{\infty} dx\, P_{\text{in}}(x) \ln P_{\text{in}}(x) \\ &= -\mathcal{I}_r + S_r^{\text{in}} \end{aligned} \tag{5.60}$$

which yields the the relation $I_r = S_r^{\text{in}} - S_r^{\text{out}} = \mathcal{I}_r$. In the same way, we obtain the relation $I_k = \mathcal{I}_k$ for the operational momentum measurement.

Therefore we have found that the information calculated as the difference between the premeasurement entropy and the average value of the postmeasurement entropy of the physical system is equal to the Shannon mutual information extracted from the result of the operational phase-space measurement. The measurement process, the entropy change, and the mutual information that we have considered are schematically shown in Fig. 2. In the figure, X, Y, and Z represent the sets of probability events in the premeasurement and postmeasurement quantum states of the physical system and in the result

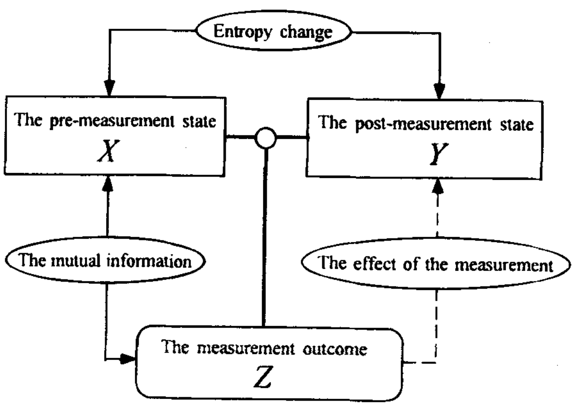

Fig. 2. Schematic representation of the measurement process, the entropy change, and the mutual information.

obtained by the measurement apparatus. Using X and Z, we can express the Shannon mutual information $\mathcal{I}$ ($= \mathcal{I}_r$ or $\mathcal{I}_k$) as

$$\mathcal{I} = H(X) - H(X|Z) = H(Z) - H(Z|X) \tag{5.61}$$

and the information calculated as the entropy change of the physical system is given by

$$I = H(X) - H(Y|Z) \tag{5.62}$$

where $H(U)$ is the Shannon entropy (the information-theoretic entropy) obtained for the set U of the probability events and $H(U|V)$ is the conditional entropy (Shannon, 1948a, b; Cover and Thomas, 1991). The equality $I = \mathcal{I}$ yields the relation $H(X|Z) = H(Y|Z)$ in our measurement model. This relation indicates that the ambiguity in our knowledge about the premeasurement state of the physical system is equal to that in our knowledge about the postmeasurement state, provided we obtain the measurement outcome for the physical system.

5.3. Simultaneous Measurement of Position and Momentum

Using the results obtained in Section 4.3, we consider the information about a physical system obtained by an operational phase-space measurement of position and momentum. In this case, to calculate the entropy and information, we need the probability distribution of position and momentum in the quantum state of the physical system, that is, the probability distribution $P(x, p)$. It is known, however, that there is no such probability distribution in quantum mechanics. Thus, to avoid this difficulty, we apply the phase-space Q-function (Husimi function) (Husimi, 1940; Kano, 1965; Mehta and Sudarshan, 1965) as the probability distribution of position and momentum for estimating the entropy and the information. The entropy calculated by means of the phase-space Q-function is called the Wehrl entropy (Wehrl, 1978, 1979).

The Wehrl entropy of the initial quantum state $\hat{\rho}$ of the physical system is given by

$$S_{rk}^{\text{in}} = -\int_{\beta \in \mathbb{R}^2} d^2\beta \; Q_{\text{in}}(\beta) \ln Q_{\text{in}}(\beta) \tag{5.63}$$

where $Q_{\text{in}}(\beta)$ is the Q-function of the initial quantum state of the physical system, namely, $Q_{\text{in}}(\beta) = \pi^{-1}\langle\beta|\hat{\rho}|\beta\rangle$ with the coherent state $|\beta\rangle$. When we obtain the measurement outcomes r and k in the operational phase-space measurement, the *conditional* Q-function of the postmeasurement state of the physical system becomes

$$Q_{\text{out}}(\beta|r, k) = \frac{1}{\pi}\frac{\langle\beta|\hat{\mathcal{L}}(r, k)\hat{\rho}|\beta\rangle}{\mathcal{W}(r, k)} \tag{5.64}$$

where the operation $\hat{\hat{\mathscr{L}}}(r, k)$ describing the state change of the physical system and the operational phase-space probability density $\mathscr{W}(r, k)$ of the measurement outcome are given by

$$\hat{\hat{\mathscr{L}}}(r, k)\hat{O} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv'\, \mathscr{K}_{p\text{-}m}(u, u'; v, v') \times e^{-i[u(\hat{x}-r)+v(\hat{p}-k)]}\hat{O}e^{i[u'(\hat{x}-r)+v'(\hat{p}-k)]} \tag{5.65}$$

$$\mathscr{W}(r, k) = \mathrm{Tr}[\hat{\hat{\mathscr{L}}}(r, k)\hat{\rho}] = \frac{1}{2\pi}\mathrm{Tr}[\hat{D}(r, k)\hat{\sigma}\hat{D}^{\dagger}(r, k)\hat{\rho}] \tag{5.66}$$

Here $\mathscr{K}_{\mathrm{p-m}}(u, u'; v, v') = {}_{ab}(u, v|\hat{\rho}_{ab}|u', v')_{ab}$ [see equation (4.38)] and the statistical operator $\hat{\sigma}$ is given by equation (4.39). Then the average value S_{rk}^{out} of the *conditional* Wehrl entropy of the physical system after the measurement is calculated as

$$S_{rk}^{\mathrm{out}} = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \mathscr{W}(r, k) S_{rk}^{\mathrm{out}}(r, k) \tag{5.67}$$

with

$$S_{rk}^{\mathrm{out}}(r, k) = -\int_{\beta\in\mathbf{R}^2} d^2\beta\, Q_{\mathrm{out}}(\beta|r, k) \ln Q_{\mathrm{out}}(\beta|r, k) \tag{5.68}$$

Using equations (5.63) and (5.67), we obtain the information I_{rk} about the physical system gained by the operational phase-space measurement of position and measurement,

$$I_{rk} = S_{rk}^{\mathrm{in}} - S_{rk}^{\mathrm{out}} \tag{5.69}$$

Then, it is easy to see from equations (5.63), (5.64), and (5.67)–(5.69) that the information I_{rk} can be expressed as

$$I_{rk} = H_{rk} - \Delta\mathscr{I}_{rk} \tag{5.70}$$

where H_{rk} is the differential entropy calculated by the operational phase-space probability density $\mathscr{W}(r, k)$ of the measurement outcome,

$$H_{rk} = -\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk\, \mathscr{W}(r, k) \ln \mathscr{W}(r, k) \tag{5.71}$$

and the quantity $\Delta\mathscr{I}_{rk}$ is given by

$$\Delta\mathscr{I}_{rk} = -\frac{1}{\pi}\int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dk \int_{\beta\in\mathbf{R}^2} d^2\beta\, \langle\beta|\hat{\hat{\mathscr{L}}}(r, k)\hat{\rho}|\beta\rangle \ln\langle\beta|\hat{\hat{\mathscr{L}}}(r, k)\hat{\rho}|\beta\rangle + \frac{1}{\pi}\int_{\beta\in\mathbf{R}^2} d^2\beta\, \langle\beta|\hat{\rho}|\beta\rangle \ln\langle\beta|\hat{\rho}|\beta\rangle \tag{5.72}$$

Using equations (5.70)–(5.72), we can estimate the information I_{rk} about the physical system extracted from the operational phase-space measurement of position and momentum, where the estimation is performed by using the Wehrl entropies.

To obtain the information I_{rk} explicitly, suppose that the measurement apparatus of position and momentum are prepared in the vacuum states,

$$\hat{\rho}_{ab} = |0\rangle_a\, {}_a\langle 0| \otimes |0\rangle_b\, {}_b\langle 0| \tag{5.73}$$

Furthermore, the measured physical system is in the coherent state $|\alpha\rangle$ with $\alpha = (q + ip)/\sqrt{2}$ before the measurement. In this case,the phase-space Q-function $Q_{\text{in}}(\beta)$ of the initial quantum state of the physical system and the operational phase-space probability density $\mathcal{W}(r, k)$ of the measurement outcome are given by

$$Q_{\text{in}}(\beta) = \frac{1}{\pi} \exp(-|\beta - \alpha|^2) \tag{5.74}$$

$$\mathcal{W}(r, k) = \frac{4}{9\pi} \exp\left(-\frac{8}{9} |\mu - \alpha|^2\right) \tag{5.75}$$

where we set $\mu = (r + ik)/\sqrt{2}$. Therefore we obtain the entropies S_{rk}^{in} and H_{rk},

$$S_{rk}^{\text{in}} = \ln(e\pi), \qquad H_{rk} = \ln\left(\frac{9\pi e}{4}\right) \tag{5.76}$$

Furthermore, the quantity $\langle\beta|\hat{\hat{\mathcal{L}}}(r, k)\hat{\rho}|\beta\rangle$ is calculated as

$$\begin{aligned} \langle\beta|\hat{\hat{\mathcal{L}}}(r, k)\hat{\rho}|\beta\rangle &= \frac{1}{4\pi^3} \left| \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, e^{-1/2(u^2+v^2)+i(ur-uk)} \langle\beta| e^{-i(u\hat{x}-v\hat{p})}|\alpha\rangle \right|^2 \\ &= \frac{1}{\pi^3} \left| \int_{z\in \mathbf{R}^2} d^2z\, e^{-|z|^2+\mu^* z-\mu z^*} \langle\beta|\hat{D}(z)|\alpha\rangle \right|^2 \\ &= \frac{4}{9\pi} \exp\left(-\frac{8}{9} |\mu - \alpha|^2 - \left|\beta - \frac{\alpha + 2\mu}{3}\right|^2\right) \end{aligned} \tag{5.77}$$

which yields $\Delta\mathcal{I}_{rk} = \ln(9\pi e/4)$. Therefore it is found from equations (5.70) and (5.76) that the information I_{rk} is estimated to be zero. This result indicates that when we measure the physical system in the coherent state with the measurement apparatus prepared in the vacuum states and we estimate the information gain by means of the Wehrl entropies, we cannot obtain any information about the system.

Next we consider the case when we know only the average value $\overline{m}$ of the photon number of the physical system. According to the maximum-entropy principle (Jaynes, 1957a, b), the quantum state of the physical system is described by the thermal state given by (5.53). In this case, the Q-function $Q_{\text{in}}(\beta)$ of the initial quantum state of the physical system and the operational phase-space probability density $\mathcal{W}(r, k)$ of the measurement outcome are given by

$$Q_{\text{in}}(\beta) = \frac{1}{\pi(1 + \overline{m})} \exp\left(-\frac{|\beta|^2}{1 + \overline{m}}\right) \tag{5.78}$$

$$\mathcal{W}(r, k) = \frac{4}{\pi(9 + 8\overline{m})} \exp\left(-\frac{8|\mu|^2}{9 + 8\overline{m}}\right) \tag{5.79}$$

which yields the entropy

$$S_{rk}^{\text{in}} = \ln[\pi e(1 + \overline{m})], \qquad H_{rk} = \ln\left[\frac{\pi e(9 + 8\overline{m})}{4}\right] \tag{5.80}$$

Furthermore, the quantity $\langle\beta|\hat{\hat{\mathcal{L}}}(r, k)\hat{\rho}|\beta\rangle$ is calculated from (5.77),

$$\begin{aligned}
&\langle\beta|\hat{\hat{\mathcal{L}}}(r, k)\hat{\rho}|\beta\rangle \\
&= \frac{1}{\pi\overline{m}} \int_{\alpha\in\mathbf{R}^2} d^2\alpha \exp\left(-\frac{|\alpha|^2}{\overline{m}}\right) \frac{4}{9\pi} \\
&\quad \times \exp\left(-\frac{8}{9}|\mu - \alpha|^2 - \left|\beta - \frac{\alpha + 2\mu}{3}\right|^2\right) \\
&= \frac{4}{9\pi(1 + \overline{m})} \exp\left[-\frac{9 + 8\overline{m}}{9(1 + \overline{m})}\left|\beta - \frac{2(3 + 4\overline{m})}{9 + 8\overline{m}}\mu\right|^2 - \frac{8}{9 + 8\overline{m}}|\mu|^2\right]
\end{aligned} \tag{5.81}$$

from which we can obtain $\Delta\mathcal{I}_{rk} = \ln(9\pi e/4)$. In this equation, we have used the P-representation of the thermal state of the physical system. Therefore, it is found from equations (5.70) and (5.80) that the information I_{rk} about the physical system is given by

$$I_{rk} = \ln(1 + \tfrac{8}{9}\overline{m}) \tag{5.82}$$

If the average photon number of the physical system is sufficiently large ($\hat{m} >> 1$), we find that $I_{rk} \approx H_{rk} \approx \ln \hat{m}$. Of course, if $\hat{m} = 0$, we cannot obtain any information about the physical system ($I_{rk} = 0$).

6. SUMMARY

In this paper, we have consider the state change, quantum probability, and information gain in an operational phase-space measurement. For this purpose, we have obtained the operational POVM which yields the quantum probability of the measurement outcome and the operation (or the complete positive instrument) which describes the state change of the measured physical system caused by the effect of the operational phase-space measurement. The properties of the operational POVM and marginal POVM and their Naimark extensions have been investigated. It has been found that the Naimark extensions are expressed in terms of the relative-position states and the relative-momentum states in the extended Hilbert space. Furthermore, it has been shown that the observable quantities measured in the operational phase-space measurement are given by fuzzy or unsharp observables which are not necessarily Hermitian operators due to the quantum noise of the measurement apparatus. The state change of the physical system measured in the operational phase-space measurement is described by an operation which has been obtained explicitly for the position measurement, the momentum measurement, and the simultaneous measurement of position and momentum. In this case, we have assumed the standard models for the interaction Hamiltonians between the measured physical system and the measurement apparatus. Using the results, we have investigated the information about the physical system gained in the operational phase-space measurement. Then we found that the average value of the entropy change of the physical system is equal to the Shannon mutual information extracted from the outcomes exhibited by the measurement apparatus.

Finally, investigations of the information about a measured physical system and the state change caused by the effect of a quantum measurement provide the basis of the quantum mechanical phenomena in the quantum communication and information theory (Bendjaballah *et al.*, 1991; Belavkin *et al.*, 1995; Hirota *et al.*, 1997). For instance, extracting information from a received quantum-state signal is nothing but an operational quantum measurement. Eavesdropping for a quantum communication system causes the state change of the signal that carries information.

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